Problem Sheet #9

Symplectic geometry. 2024 Winter Term. Heidelberg University
Course taught by J.-Pr. Agustín Moreno*

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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises. **You are encouraged to work in pairs!**
Deadline: *January 10, 2024.*

Problems

Exercise 1. Let $S \subseteq (M, \omega)$ be a compact hypersurface and assume that S bounds a symplectic manifold B, i.e. $\partial B = S$. Let S_ϵ be a parametrized family of hypersurfaces modeled on S and denote by B_ϵ the symplectic manifold bounded by S_ϵ. We assume the parametrization is such that $\epsilon \leq \epsilon' \implies B_\epsilon \subseteq B_{\epsilon'}$. Recall that if $C(\epsilon) = c_0(B_\epsilon, \omega)$, the surface S_{ϵ^*} is called of c_0-Lispchitz type if there exist $\mu, L > 0$ such that

$$C(\epsilon) \leq C(\epsilon^*) + L(\epsilon - \epsilon^*)$$

for all $\epsilon^* \leq \epsilon \leq \epsilon^* + \mu$.

1. Show that the c_0-Lipschitz condition does not depend on the choice of parametrized family on S_{ϵ^*} but only on $S_{\epsilon*}$.
2. Let S be the hypersurface described at the beginning of the exercise and assume that there exists a vector field X defined in neighborhood of S that is transverse to S and satisfies $\mathcal{L}_X\omega = \omega$ (X is called a *Liouville vector field*). Show that S is of c_0-Lipschitz type.

Exercise 2. (Introduction to Liouville domains)

1. Let M be a closed even-dimensional manifold. Show that it admits no exact symplectic structure.

Let $(W, \omega = \mathrm{d}\lambda)$ be a compact, exact symplectic manifold with boundary. We call it a **Liouville domain** if the boundary ∂W is of restricted contact-type, i.e if $\lambda|_{\partial W}$ is a contact form.

2. Restate the 'restricted contact-type' condition in terms of the Liouville vector field.

*For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or ruscelli.francesco1@gmail.com

3. Show that the closed disc $\mathbb{D}^{2n} \subset (\mathbb{R}^{2n}, \omega_0 = \mathrm{d}\lambda_0)$ is a Liouville domain, with

$$\lambda_0 = \frac{1}{2} \sum_i q_i \mathrm{d}p_i - p_i \mathrm{d}q_i$$

Let Q be a manifold which we assume, for simplicity, to be embedded into some $\mathbb{R}^N$, with ambient inner product $\langle \cdot, \cdot \rangle$. Then, the cotangent bundle of Q is defined as:

$$T^\star Q = \left\{ (q,p) \in T^\star \mathbb{R}^N \mid q \in Q, \langle q, p \rangle = 0 \right\} \tag{1}$$

and we further make the two definitions:

$$\mathbb{D}^\star Q = \left\{ (q,p) \in T^\star \mathbb{R}^N \mid q \in Q, \langle q, p \rangle = 0, \|p\| \leq 1 \right\} \tag{2}$$

$$\mathbb{S}^\star Q = \left\{ (q,p) \in T^\star \mathbb{R}^N \mid q \in Q, \langle q, p \rangle = 0, \|p\| = 1 \right\} \tag{3}$$

which are submanifolds of $T^\star Q$, respectively called the **disc cotangent bundle** and **unit cotangent bundle**.

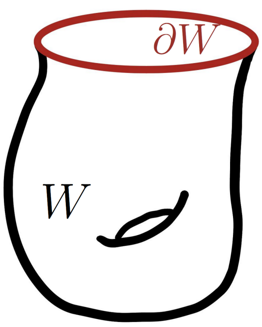
3. Prove that $\partial(\mathbb{D}^\star Q) = \mathbb{S}^\star Q$, as manifolds.

4. Show that, when endowed with the natural symplectic structure inherited from $T^\star Q$, $\mathbb{D}^\star Q$ is a Liouville domain.

Exercise 3. In practice, if one wants to do dynamics on a Liouville domain, the boundary often poses problems, as our usual tools from symplectic geometry are not all meant to deal with such cases (we often work on closed manifolds, i.e compact without boundary). As it happens, it is easier to get away with lifting the compactness assumption, than with having a boundary. Therefore, we will employ a scheme called **Liouville extension**, which will turn our Liouville domain into a non-compact manifold without boundary. Our notation is as in the previous exercise.

Recall from lectures that since the Liouville vector field V is transverse to the boundary, we have $(\phi_V^t)^\star \lambda = e^t \lambda$ for $t \in (-\epsilon, 0]$. Write $r = e^t$.

1. Define $\widehat{W} := W \cup_{\partial W} [1, +\infty) \times \partial W$. We call this the Liouville extension of W. Draw a picture of this extension process in the cases $W = \mathbb{D}^{2n}$, $W = \mathbb{D}^\star \mathbb{S}^1$ (*Note:* for the latter, assume $\mathbb{S}^1$ is embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a circle of radius > 1), and for the following manifold:



We often call this extension $\widehat{W}$ a Liouville manifold. It can be endowed with an exact symplectic structure given by $\hat{\omega} = \mathrm{d}(r\lambda)$ (which is a smooth extension of $\omega = \mathrm{d}\lambda$).

Now, the easiest type of Hamiltonians to work with on a Liouville manifold are those which are linear at infinity, i.e:

$$H : \widehat{W} \to \mathbb{R} \text{ is such that } \exists r_0 \geq 1 \text{ s.t } H = H(r) = ar + b \text{ on } [r_0, +\infty) \times \partial W \tag{4}$$

for some constants $a > 0$, $b \in \mathbb{R}$.

We define an almost complex structure J on $\widehat{W}$ as follows: First, choose an almost complex structure J_ξ on $\xi := \ker \lambda|_{\partial W}$ (the natural contact structure on ∂W). And extend it to $[1, +\infty) \times \partial W$ by setting:

$$J\partial_r = \mathcal{R}$$

where $\mathcal{R}$ is the Reeb vector field on $(\partial W, \lambda|_{\partial W})$. (*A priori*, it is only defined on ∂W, but we can translate it to every slice $\{r\} \times \partial W$, for $r > 1$). We then smoothly extend this J to the interior of W in an arbitrary way.

2. Let $H : \widehat{W} \to \mathbb{R}$ satisfy (4). Compute X_H at infinity.
3. Deduce that, for $r \geq r_0$, periodic orbits are constrained to slices $\{r\} \times \partial W$ which are "parallel to the boundary".

Let $\alpha := \lambda_{\partial W}$ denote the restricted contact form on ∂W, and define

$$\operatorname{spec} \alpha := \{T > 0 \mid T \text{ is the period of a Reeb orbit}\}$$

And we make the assumption that the slope a from (4) satisfies $a \notin \operatorname{spec} \alpha$.

4. Show that the period 1 periodic orbits of H are all contained in a compact region.

Exercise† 4. (Morse Lemma) Let $f \colon M \to \mathbb{R}$ be a Morse function and $p \in M$ a critical point. Prove that there exists a chart $\varphi \colon U \to \mathbb{R}^n$ around p such that

$$f\varphi^{-1}(x_1, \dots, x_n) = f(p) - (x_1^2 + \dots + x_k^2) + (x_{k+1}^2 + \dots x_n^2)$$

for some $0 \leq k \leq n$.
Hint: use Moser's trick.

Exercise† 5. (Some Morse homology) Consider a torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$, which is slightly tilted (it's standing almost vertically, but not quite). Write $f : \mathbb{T}^2 \to \mathbb{R}$ the restriction of the height function $(x, y, z) \mapsto z$ to the torus.

1. Draw the critical points of f, along with their Morse indices (you don't need to compute them explicitly, but justify your answer).
2. Verify explicitly that, in this case, $HM_*(\mathbb{T}^2, f; \mathbb{Z}_2) \cong H_*(\mathbb{T}^2; \mathbb{Z}_2)$, where HM_* denotes Morse homology, and H_* singular homology.