Problem Sheet #8

Symplectic geometry. 2024 Winter Term. Heidelberg University
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December 6, 2024

Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises. **You are encouraged to work in pairs!**
Deadline: Friday Dec. 13 2024. (Note: this is the last due submission for the year 2024. A final, Christmas problem sheet will be uploaded after the Dec. 13 class, due in January).

Problems

Exercise 1.

1. Let g be a scalar product on $\mathbb{R}^{2n}$ and consider the ellipsoid
$$E(g) = \{v \in \mathbb{R}^{2n} \mid \ g(v,v) < 1\}.$$
Show that there exists $A \in \mathrm{Sp}(2n)$ and $r = (r_1, \dots, r_n)$ with $0 < r_1 \le \dots \le r_n$ such that $A\big(E(g)\big) = E(r)$, where
$$E(r) = \{(x,y) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i^2} < 1\}.$$
Hint: you can use the fact that, if (V, ω) is a symplectic vector space and $\langle, \rangle$ is a scalar product, there exists a symplectic basis $\{e_i, f_i\}$ that is orthogonal with respect to $\langle, \rangle$. Furthermore, this basis can also be chosen to satisfy $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle$ for all i.

2. Show that the numbers $r_1, \dots, r_n$ are uniquely determined by $E(g)$.
Hint: suppose that $E(r)$ and $E(s)$ are related by an element $A \in \mathrm{Sp}(2n)$. Show that the matrices $J_0 \mathrm{diag}(\frac{1}{r_1^2}, \dots, \frac{1}{r_n^2})$ and $J_0 \mathrm{diag}(\frac{1}{s_1^2}, \dots, \frac{1}{s_n^2})$ are similar.

Exercise 2. (Isoperimetric inequality) Let (V, ω) be a symplectic vector space and let $J \in \mathcal{J}(V, \omega)$ be an ω-compatible linear complex structure. Denote by $\|v\|^2 = \omega(v, Jv)$ for $v \in V$. Consider a smooth loop $\gamma \colon \mathbb{R}/\mathbb{Z} \to V$ and define
$$A(\gamma) = \frac{1}{2} \int_0^1 \omega(\dot{\gamma}(t), \gamma(t))\, \mathrm{d}t,$$
$$E(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|\, \mathrm{d}t,$$
$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|\, \mathrm{d}t,$$

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which are the (linear) **symplectic action**, the **energy** and the **length** of γ respectively. Prove that

$$|A(\gamma)| \leq \frac{1}{4\pi}L(\gamma)^2 \leq \frac{1}{2\pi}E(\gamma).$$

If γ is nonconstant, prove that $|A(\gamma)| = \frac{1}{2\pi}E(\gamma)$ if and only if the image of γ is a circle.
Hint: identify $(V, \omega, J) = (\mathbb{C}^n, \omega_0, J_0)$ and write γ as a Fourier series $\gamma(t) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi J_0 kt}$ with $a_k \in \mathbb{C}^n$ for all $k \in \mathbb{Z}$. Prove that

$$A(\gamma) = -\pi \sum_{k\in\mathbb{Z}} k \left\| a_k \right\|^2,$$
$$E(\gamma) = 2\pi^2 \sum_{k\in\mathbb{Z}} k^2 \left\| a_k \right\|^2$$

and deduce that $|A(\gamma)| \leq \frac{1}{2\pi}E(\gamma)$. Approximate γ by immersed loops and reparametrize by arc length.

Exercise 3. (Principle of Least Action) Let $(M, \omega = \mathrm{d}\lambda)$ be a compact, exact symplectic manifold, with an almost complex structure J, and consider $\mathscr{P} := \mathcal{C}^\infty(\mathbb{S}^1, M)$, the space of smooth loops in M. Let H be a (possibly time-dependent) Hamiltonian on M. Then, inspired by classical physics, we define the action functional:

$$\mathcal{A}_H : \mathscr{P} \to \mathbb{R} : x \longmapsto -\int_{\mathbb{S}^1} x^\star\lambda + \int_{\mathbb{S}^1} H \circ x \tag{1}$$

The goal of this exercise is to compute the derivative of $\mathcal{A}_H$. The subsequent analysis will take place in $\mathcal{C}^\infty(\mathbb{S}^1, M)$, which is technically a(n infinite-dimensional) Banach manifold; but for the purposes of this exercise, you may assume objects behave like on finite-dimensional manifolds.

1. Let x_s be a path in $\mathscr{P}$, and $\zeta := (\mathrm{d}/\mathrm{d}s)x_s|_{s=0}$. ($\zeta$ is a tangent vector to a *loop* x in M. Therefore, it is a vector field $\zeta = \zeta(t) \in T_{x(t)}M$. Formally, one can view it as a section of the bundle $x^\star TM \twoheadrightarrow [0, 1]$). Show that:
$$\mathrm{d}\mathcal{A}_H(x)\zeta = -\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_{\mathbb{S}^1} x_s^\star\lambda + \int_{\mathbb{S}^1} \mathrm{d}H\big(\zeta(t)\big)\mathrm{d}t$$

2. Show that $\displaystyle\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} x_s^\star\lambda = x^\star\mathcal{L}_\zeta\lambda$, where $\mathcal{L}$ denotes the Lie derivative.

3. Using the sign convention $i_{X_H}\omega = -\mathrm{d}H$, show that:
$$\mathrm{d}\mathcal{A}_H(x)\zeta = \int_{\mathbb{S}^1} \mathrm{d}\lambda(\dot{x}(t) - X_H(x(t)), \zeta(t))\mathrm{d}t \tag{2}$$

4. State and prove a *"Principle of Least Action"* for periodic orbits.

Remark. If you are familiar with *Morse theory*, then the previous exercise may have given you ideas. Morse theory is a homological construction meant to detect critical points of functions $f : M \to \mathbb{R}$ on (finite-dimensional) manifolds. It does so by connecting these critical points by trajectories of the flow of $-\nabla f$ (one can prove that such trajectories have to end in critical points f); and then using these trajectories to define a "differential" (an algebraic map between formal sums of critical points), and then a homology theory.

In our case, if we could do the same with $\mathcal{A}_H : \mathcal{C}^\infty(\mathbb{S}^1, M) \to \mathbb{R}$, we could get a homology theory which records periodic orbits of our Hamiltonian flow (from a physics point of view: trajectories of our physical system). The obstruction, however, is that $\mathcal{C}^\infty(\mathbb{S}^1, M)$ is infinite-dimensional, making the constructions much more technical. Re-proving the statements from Morse theory on such infinite-dimensional manifolds is the essence of *Floer theory*; which has become a major subject in symplectic topology.

The next (bonus) exercise is a first step in this direction, which follows from Exercise 3. It aims to explain which objects we will use to connect critical points of $\mathcal{A}_H$.

Exercise† 4.

We are in the same set-up as exercise 1, with $\mathscr{P} := \mathcal{C}^\infty(\mathbb{S}^1, M)$, and $\mathcal{A}_H$ defined as in (1). You may use without proof the fact that:

$$\forall \zeta_1, \zeta_2 \in T_x\mathscr{P} : \ \langle \zeta_1, \zeta_2 \rangle := \int_{\mathbb{S}^1} g\big(\zeta_1(t), \zeta_2(t)\big)\mathrm{d}t = \int_{\mathbb{S}^1} \omega\big(\zeta_1(t), J_t\zeta_2(t)\big)\mathrm{d}t \tag{3}$$

defines an L^2-metric on $\mathscr{P}$, and that the gradient ∇ w.r.t to it is defined as usual.

1. Let $u = u(s,t)$ denote a cylinder $\mathbb{R} \times \mathbb{S}^1$. Show that the equation:

$$\frac{\partial u}{\partial s} = -\nabla \mathcal{A}_H\big(u(s)\big)$$

can be re-written:

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H = 0 \tag{4}$$

This is called the **Floer equation**. Given a solution u, we define its **energy**:

$$E(u) := \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial s}\right|^2 \mathrm{d}s \wedge \mathrm{d}t \tag{5}$$

2. Show that $E(u) = 0 \iff u \equiv x$ where x is such that $\mathrm{d}\mathcal{A}_H(x) \equiv 0$ (in other words, $E(u) = 0$ iff u is constantly equal to a periodic orbit of the Hamiltonian flow).

3. Show that $E(u)$ can be re-written:

$$E(u) = \int_{\mathbb{R}\times[0,1]} \omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_H\right)\mathrm{d}s \wedge \mathrm{d}t$$

4. Prove the following proposition:

   Proposition 0.1. *Let $u : \mathbb{R} \times \mathbb{S}^1 \to M$ be a smooth cylinder which solves the Floer equation (4), and such that:*

$$\lim_{s\to-\infty} u(s,t) = x(t), \quad \lim_{t\to+\infty} u(s,t) = y(t)$$

   where x and y are periodic orbits of the flow of H. Then, we have:

$$E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y) \tag{6}$$