Problem Sheet #7

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises. **You are strongly encouraged to work in pairs!**

Problems

Exercise 1. (Contact squeezing) Darboux's theorem implies that you can always symplectically embed a small enough ball in any symplectic manifold. For contact manifolds the situation is even better (or worse, depending on how you view things).
Let (M^{2n+1}, ξ) be a contact manifold. Show that for any $r > 0$, you can find a contact embedding $\big(B(r), \ker \alpha_0\big) \to (M, \xi)$, where α_0 is the standard contact structure on $\mathbb{R}^{2n+1}$.

Exercise 2. Let M, N be manifolds of the same dimension and assume M is compact. Let $f\colon M \to N$ be a smooth function and $q \in N$ a regular value.

1. Show that $\# f^{-1}(q)$ is finite, where $\# A$ denotes the cardinality of the set A.
2. Show that the map $\mathcal{R}_f \to \mathbb{N}\colon q \mapsto \# f^{-1}(q)$ is locally constant, where $\mathcal{R}_f \subseteq N$ is the set of regular values of f.

Exercise† 3. (Some functional analysis) Let X, Y be topological spaces. If $K \subseteq X$ is compact and $U \subseteq Y$ is open, define

$$S(K, U) = \{f \in C(X, Y) \mid f(K) \subseteq U\}.$$

These sets form a subbasis for a topology on $C(X, Y)$, which is called *compact open topology*.

1. Show that if X is compact and Y is a metric space with metric d_Y, the compact open topology is defined by the metric

$$d(f, g) = \sup_{x \in X} d_Y\big(f(x), g(x)\big).$$

2. Show that if X is σ-compact (i.e. it is the union of a countable set of compact subsets), then the compact open topology is metrizable. In particular, continuous functions between manifolds carry a metrizable topology, often called the C^0-topology.

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Exercise 4. Let $Q(r) = (0, r)^{2n} \subseteq (\mathbb{R}^{2n}, \omega_0)$ for $r > 0$ and define

$$\gamma(M, \omega) = \sup\{r^2 \mid \exists \text{ symplectic embedding } \varphi\colon Q(r) \to M\}.$$

Show that γ is monotone, conformal and nontrivial, in the sense that $\gamma(Z(1), \omega_0) < \infty$ and $\gamma(B(1), \omega_0) > 0$ (it is a capacity in a weaker sense, not in the sense we defined in class).

Exercise 5. Assume $h\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a homemomorphism so that $\gamma(h(U)) = \gamma(U)$ for all open sets U. Let μ be the Lebesgue measure on $\mathbb{R}^{2n}$.

1. Show that $\mu(Q(r)) = \gamma((Q(r)))^n$ and $\mu(U) \geq \gamma(U)^n$ for all open sets $U \subseteq \mathbb{R}^{2n}$.
2. Show that $\mu(Q(r)) \leq \mu(h(Q(r)))$.
3. Show that $\mu(U) \leq \mu(h(U))$ for all open sets $U \subseteq \mathbb{R}^{2n}$.
4. Conclude that h preserves the Lebesgue measure μ.