Problem Sheet #6

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises.

Please, hand in this property before **Friday Nov. 15** (either in person at the exercise class, or by email at alimoge@mathi.uni-heidelberg.de)

Problems

Exercise 1. Let

$$B(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \ ||x||^2 + ||y||^2 \leq r^2\},$$
$$Z(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \ |x_1|^2 + |y_1|^2 \leq r^2\},$$
$$Z_{iso}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \ |x_1|^2 + |x_2|^2 \leq r^2\}.$$

Compute their symplectic capacities. Here we are using *normalized* capacities, i.e. both the ball and the cylinder of radius 1 have capacity π.

Exercise 2. Let $B(r) \subseteq \mathbb{R}^2$ be the open disk of radius r.

- Show that there exists a volume preserving diffeomorphism

$$\psi \colon B(1) \times B(1) \to B(r) \times B\left(\frac{1}{r}\right)$$

  for any $r > 0$.

- Let c be a symplectic capacity on $\mathbb{R}^4$. Show that

$$c\left(B(r) \times B\left(\frac{1}{r}\right), \omega_0\right) \to 0$$

  as $r \to 0$.

Exercise 3.

- Let $A, B \subseteq \mathbb{R}^n$ be closed subsets and define

$$d_H(A, B) = \sup_{x \in A} d(x, B) + \sup_{y \in B} d(A, y).$$

  Show that d_H defines a metric on the set of closed subsets of $\mathbb{R}^n$. It is called the *Hausdorff metric.*

For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or ruscelli.francesco1@gmail.com

- Let c be a capacity on $\mathbb{R}^{2n}$. Prove that the restriction of c to compact convex sets is continuous with respect to the Hausdorff metric. **Hint:** to prove continuity at A, distinguish the cases $\mathrm{int}(A) = \emptyset$ and $\mathrm{int}(A) \neq \emptyset$. In the first case, show that A is contained in some hyperplane and show that it can be embedded in the cylinders $Z(\delta)$ for every $\delta > 0$ via a linear symplectic map. In the second case, if $0 \in \mathrm{int}(A)$, show that for every $\lambda > 1$ there is a $\delta > 0$ such that for every compact convex $B \subseteq \mathbb{R}^{2n}$,
$$d(A, B) < \delta \implies \frac{1}{\lambda} A < B < \lambda A.$$