Problem Sheet #5

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises.

Please, hand in this property before **Friday Nov. 15** (either in person at the exercise class, or by email at alimoge@mathi.uni-heidelberg.de)

Problems

We recall the Weinstein neighbourhood theorem: Let (W, Ω) be a compact symplectic manifold, and $L \subset W$ a compact Lagrangian. Then there exists a neighbourhood $\mathcal{U}$ of L in W which is symplectomorphic to a neighbourhood of the zero section in $(T^\star L, \omega_{\text{std}})$, where we identify the zero section with $L \hookrightarrow T^\star L$.

Exercise 1. Let (M, ω) be a compact symplectic manifold with $H^1_{\text{dR}}(M) = 0$ (i.e all closed 1-forms are exact), and which we embed in $T^\star M$ as the zero section.

1. Let σ be a 1-form on M. Recall under which conditions $\text{Graph}(\sigma)$ is Lagrangian in $T^\star M$.
2. Show that if σ is sufficiently $\mathcal{C}^1$ close to zero, then $\text{Graph}(\sigma)$ intersects the zero section in $T^\star M$ at least twice.

Exercise 2. Let (M, ω) be a compact symplectic manifold with $H^1_{\text{dr}}(M) = 0$ and $f : M \to M$ a symplectomorphism.

1. Show that $\text{Graph}(f)$ is Lagrangian $(M \times M, \omega \ominus \omega)$, where $\omega \ominus \omega := (\omega, -\omega)$.
2. Provided that f is sufficiently $\mathcal{C}^1$ close to the identity, explain how one can identify $\text{Graph}(f) \subset M \times M$ with $\text{Graph}(\eta) \subset T^\star M$ for some closed 1-form η on M.
3. Deduce that if f is a symplectomorphism which is sufficiently $\mathcal{C}^1$ close to the identity, then it has at least two fixed points.

This result can be refined by working on specific manifolds. For example, if $M = \mathbb{S}^2$, then *every* symplectomorphism has at least two fixed points. And since $\dim \mathbb{S}^2 = 2$, this can be rephrased as saying that every area-preserving diffeomorphism of $\mathbb{S}^2$ has at least two fixed points. And both of these conditions are essential!

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4. Note that when we say "area-preserving", we really mean "area-form preserving"; so that our map not only preserves the absolute value of the area, but also the orientation. Show that, if we drop the second condition, then one can find diffeomorphisms of $\mathbb{S}^2$ with **zero** fixed point.

5. Find an example of diffeomorphism on $\mathbb{S}^2$ with exactly **one** fixed point.

   Hint: so you want your group to act freely on $\mathbb{S}^2 \backslash \{\mathrm{pt}\}$. What is this diffeomorphic to?

Exercise 3. Consider $\mathbb{R}^2$ with coordinates (q, p), and the action of $\mathbb{R}$ on $\mathbb{R}^2$ consisting of translation in the q-coordinate. Show that its moment map is given by p, the standard (linear) momentum from classical physics.

Exercise 4. Let SO_3 denote the Lie group of rotations in $\mathbb{R}^3$, and recall that:

$$\mathfrak{so}_3 = \mathrm{Lie}(SO_3) = \left\{ A \in \mathcal{M}_3(\mathbb{R}) \mid A + A^t = 0 \right\}$$

1. Show that there is an isomorphism of Lie algebras $(\mathfrak{so}_3, [\cdot, \cdot]) \longrightarrow (\mathbb{R}^3, \times)$ given by:

$$\psi : \mathfrak{so}_3 \longrightarrow \mathbb{R}^3 : \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

2. Compute the infinitesimal generator of the standard action of SO_3 on $\mathbb{R}^3$.

3. Deduce that $\mu = \vec{q} \times \vec{p}$ (the physical angular momentum) is a moment map for the action.