Problem Sheet #10

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises. **You are encouraged to work in pairs!**

The material/definitions in this exercise sheet are not examinable. However, the techniques used to solve the exercises are, and should all already be familiar from the course.

Deadline: January 24, 2024.

Problems

We recall that Hamiltonian Floer theory is concerned with period 1 periodic orbits of a Hamiltonian $H : M \to \mathbb{R}$, where M is a closed, **symplectically aspherical** symplectic manifold.

Exercise 1. A symplectic manifold (M, ω) is called symplectically aspherical if every sphere has symplectic area zero, i.e

$$\forall u : \mathbb{S}^2 \to M : \int_{\mathbb{S}^2} u^\star \omega = 0 \tag{1}$$

1. Show that Liouville domains (defined in the previous sheet) are symplectically aspherical.

Assume $(M, \omega = \mathrm{d}\lambda)$ is a Liouville domain, and let $L \subset M$ be a Lagrangian submanifold. We call L exact if $\exists f : L \to \mathbb{R}$ such that $\lambda|_L = \mathrm{d}f$.

2. Show that, if L is exact, then (1) also holds for discs $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ (i.e, discs with Lagrangian boundary).

3. Let $Q = \mathbb{S}^2 = \{(\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3 \mid \xi_0^2 + \xi_1^2 + \xi_2^2 = 1\}$, and define $\mathbb{D}^\star Q, \mathbb{S}^\star Q \subset T^\star Q$, like in the previous problem sheet. Consider the submanifold:

$$R := \{(x, y, z) \in \mathbb{R}^3 \mid \xi_0^2 + \xi_2^2 = 1\} \cong \mathbb{S}^1$$

Compute its co-normal bundle (i.e, the dual bundle to its normal bundle) and show that its intersection with $\mathbb{D}^\star\mathbb{S}^2$ is exact Lagrangian.

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Lagrangian Floer theory is constructed in a very similar way as Hamiltonian Floer theory, except instead of trying to detect periodic orbits of a Hamiltonian, one considers two Lagrangians L_1, L_2 in a symplectic manifold, and looks at all the intersections $L_1 \cap L_2$.

Exercise 2. Show that Hamiltonian Floer theory can be viewed as a special case of Lagrangian Floer theory.

Hint: recall that a periodic orbit of a Hamiltonian $H : M \to \mathbb{R}$ is a fixed point of its time-1 map $\phi = \phi_H^{t=1}$(And remember Problem Sheet 5!)

Exercise† 3. Let (M, ω) be compact, symplectically aspherical, and $H : M \to \mathbb{R}$ be a Hamiltonian. In the vocabulary of Hamiltonian Floer theory, a 1-periodic orbit $\gamma : \mathbb{S}^1 \to M$ of H is called **degenerate** if $D\phi|_{\gamma(0)}$ does not have 1 as an eigenvalue.

Show that, if we view Hamiltonian Floer theory as a Lagrangian Floer theory, then the notion of degeneracy can be rephrased as non-transversality of intersections.

Hint: it is enough to work locally, so you can assume $M = \mathbb{R}^n$. Then, use the implicit function theorem to write both your Lagrangians in a nice easy form.