Problem Sheet #1

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises.

Note: in this sheet (and in life), all manifolds/functions/differential forms are smooth.

Problems

Exercise 1. Let V be a finite-dimensional (real) vector space, and $V^\star := \mathrm{End}(V, \mathbb{R})$. Recall that an element $\eta \in V^\star$ is called a 1-form, and that given k 1-forms $\eta_1, \ldots, \eta_k$, we can define their exterior product:

$$\forall v_1, \ldots, v_k \in V : \big(\eta_1 \wedge \cdots \wedge \eta_k\big)(v_1, \ldots, v_k) := \det(\eta_1(v_1), \ldots, \eta_k(v_k))$$

A k-form is a linear combination of such exterior products of k 1-forms.

1. Show that the space $\Lambda^k V$ of k-forms on V is a vector space. Given a basis for V, write down a basis for $\Lambda^k V$, and compute $\dim \Lambda^k V$.
2. Write down the canonical basis for $\Lambda^n V$, where $n := \dim V$. (This should not depend on your chosen basis for V).

A 2-form is called **non-degenerate** if it induces an isomorphism $V \xrightarrow{\cong} V^\star$.

3. Show that the existence of a non-degenerate 2-form implies V is even-dimensional.
4. Let $V \cong \mathbb{R}^{2n}$ with basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$, and with standard Euclidean inner product g. Show that the map J defined by:

$$Jv_i = w_i, \;\; Jw_i = -v_i$$

is an almost complex structure on V. (i.e $J^2 = -id$).

5. Compute $\omega := g(\cdot, J\cdot)$, and show that it is a non-degenerate 2-form on V.

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Exercise 2. Let M be a manifold.

1. Define the space of differential k-forms $\Omega^k(M)$. Show that it is a vector space.
2. Give a definition of the exterior (or de Rham) derivative d.

Recall that $\omega \in \Omega^k(M)$ is called *closed* if $\mathrm{d}\omega = 0$, and *exact* if $\exists \eta \in \Omega^{k-1}(M)$ s.t $\omega = \mathrm{d}\eta$. And recall Stokes' theorem, which says that given M a manifold with boundary, we have:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \tag{1}$$

3. Let ω be a closed 2-form on M, and $S_1 \subset M$ an embedded surface. Given a flow $\phi^t : M \to M$, write $S_2 := \phi^1(S_1)$. Show that:

$$\int_{S_1} \omega = \int_{S_2} \omega$$

Exercise 3. Let M be a compact, orientable, n-dimensional manifold. We denote Z^k the space of closed k-forms on M, and B^k the space of exact forms.

1. Show that B_k and Z_k are sub-vector spaces of $\Omega^k(M)$.
2. Show $B_k \subset Z_k$, but that in general, this is not an equality.
3. Show that $H^k := Z^k/B^k$ is a vector space, and that the map:

$$\int_M : H^n(M) \longrightarrow \mathbb{R}$$

is a (well-defined) vector space isomorphism, where $\int_M$ denotes integration.

Exercise 4. Let M be a manifold. Inspired by Exercise 1., we call a 2-form non-degenerate if it induces an isomorphism between the tangent and cotangent spaces of M at each point. And we call a 2-form **symplectic** if it is closed and non-degenerate.

1. Show that a surface S is symplectic iff it is orientable.
2. Show that the symplectic structure on S is unique (in a reasonable sense of the term, which you should explain).

Exercise 5. Recall that a Lie group is a group G that is endowed with a smooth structure such that the group operations are smooth. A vector field X on G is called *left-invariant* if $(L_g)_* X = X$ for all $g \in G$, where

$$\begin{aligned} L_g \colon G &\to G \\ h &\mapsto gh \end{aligned}$$

is left multiplication by g. Define the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ as the set of left-invariant vector fields on G.

- Show that there is a natural identification $\mathfrak{g} \cong T_e G$, where e is the identity element of G.
- Show that the orthogonal group

$$O(n) = \{A \in M(n, \mathbb{R}) \mid AA^t = I\}$$

  is a Lie group and compute its dimension. (Bonus question: what is its Lie algebra?)