We have seen that the Hofer-Zehnder capacity c_0 is indeed a capacity. We are now going to use this to obtain results of dynamical nature.

Consider a symplectic mfd (M, ω) and a Hamiltonian function $H \in C^\infty(M, \mathbb{R})$.

Suppose $S = H^{-1}(c)$ is a regular level set for some $c \in \mathbb{R}$. and assume S is cpt.

$\Rightarrow$ S is a ~~compact~~ closed submfd of M of codimension 1.

and $TS = (\ker dH)\big|_S$.

It is easy to see that the Hamiltonian vector field X_H is tangent to S. Indeed, $dH(X_H) = -\omega(X_H, X_H) = 0$ on S.

<u>Problem</u> Does X_H admit closed orbits on S?

First, note that the existence of closed orbits does not depend on the choice of H.

Indeed, suppose $S = \{H \equiv c\} = \{F \equiv c\}$ for two Hamiltonians $H, F \in C^\infty(M, \mathbb{R})$ with $dH, dF \neq 0$ on S.

Then, $\forall x \in S$: $\ker d_xH = \ker d_xF \Leftrightarrow$ $d_xF = \rho(x) d_xH$. for a nonvanishing smooth function ρ on S.

$\Rightarrow X_F = \rho X_H$ on S.

If φ^t is the flow of X_H on S, ψ^s is the flow of X_F on S, we have $\psi^s(x) = \varphi^t(x)$ $\forall x \in S$,

when $t = t(x,s)$ is a function determined by the ODE

$$\begin{cases} \frac{dt}{ds} = \rho(\varphi^t(x)) \\ t(x,0) = x. \end{cases}$$

$\Rightarrow X_H, X_F$ have the same flow lines and, in particular, the same periodic orbits.

Remark
There is a geometric way of viewing this problem, that also shows independence of H.

Let $S \subseteq (M^n, \omega)$ be any codimension 1 submfd.

Define $\mathcal{L}_S = \ker \omega|_S$. By nondegeneracy of ω on M and the fact that $\dim S = 2n-1$, $\mathcal{L}_S$ is a line bundle.

If S is a cpt regular level set of some $H: M \to \mathbb{R}$,

$\mathcal{L}_S(x)$ $\forall x \in S$. (as we showed earlier).

$X_H(x) \in \mathcal{L}_S(x)$

(note that the condition $dH \neq 0$ on S implies $X_H \neq 0$ on S).

$\Rightarrow \mathcal{L}_S$ is orientable (in particular trivial).

Conversely, suppose $\mathcal{L}_S \to S$ is orientable. We will construct a function $H: U \to \mathbb{R}$, where U is a nbhd of S such that $S = H^{-1}(0)$ is a regular level set.

Pick an almost complex structure on J compatible with ω.
In particular, $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ is a scalar product.

Let $N_S = \{ \xi \in T_xM \mid \langle \xi, v \rangle = 0 \ \forall v \in TS \}$ be the normal bundle of S. Note that the map

$$\begin{aligned} \mathcal{L}_S &\to N_S \\ \xi &\mapsto J\xi \end{aligned}$$ is a bundle isomorphism.

$\mathcal{L}_S$ is trivial by assumption $\Rightarrow$ N_S is trivial.
Pick a nonvanishing section $z: S \to N_S$ and define

$$\begin{aligned} \varphi: \ & S \times (-\varepsilon, \varepsilon) \to M \\ & (x,t) \longmapsto \exp_x(t z(x)) \end{aligned}$$

This is a diffeomorphism onto a nbd U of S if $\varepsilon > 0$ is small enough (we are using the fact that S is cpt).

If $$\begin{aligned} F: \ & S \times (-\varepsilon, \varepsilon) \to \mathbb{R} \\ & (x,t) \longmapsto t \end{aligned}$$, the desired Hamiltonian is given by

$$H := F\varphi^{-1} : U \to \mathbb{R}.$$

~~[illegible]~~ $\mathcal{L}_S$ is called "characteristic line bundle" of S.

Our problem of finding closed orbits of X_H on S is thus purely geometric, i.e. it is equivalent to finding an embedded circles $P \subseteq S$ such that $TP = \mathcal{L}_S|_P$. Such a circle is called a "closed characteristic".

Note that our construction provides a nbd of S that is foliated by hypersurfaces diffeomorphic to S. This prompts the following definition:

<u>Def.</u> Let S be a cpt hypersurface in (M,ω). A parametrized ~~fam~~ family of hypersurfaces ~~of~~ modeled on S is a diffeomorphism

$$\psi : S \times I \to U \subseteq M, \quad I \text{ open interval containing } 0 \in \mathbb{R}$$

such that $\psi(x,0) = x \quad \forall x \in S$.

We are going to denote ~~this~~ such a family by $(S_\varepsilon)_{\varepsilon \in I}$.

~~We have thus shown the following~~

Rephrasing our work so far, we have shown that the following statement are equivalent:

(i) $\mathcal{L}_S \to S$ is orientable

(ii) $N_S \to S$ is orientable

(iii) S is orientable

(iv) There exists a parametrized family of hypersurfaces modeled on S.

(v) $\exists\, H : \underset{M}{\underset{\cap}{U}} \to \mathbb{R}$, U nbd of S satisfying $dH \neq 0$ on S.

Our ~~search~~ search for closed characteristics starts with the following theorem by Hofer and Zehnder.

<u>Thm [Hofer-Zehnder]</u>

Let S be a cpt hypersurface and (S_ε) a parametrized family of surfaces modeled on S. Let $P(S_\varepsilon)$ be the set of closed characteristics on S_ε. Then, if $c_0(U,\omega) < \infty$, there exists a dense set $\Sigma \subseteq I$ such that $P(S_\varepsilon) \neq \emptyset \quad \forall \varepsilon \in \Sigma$.

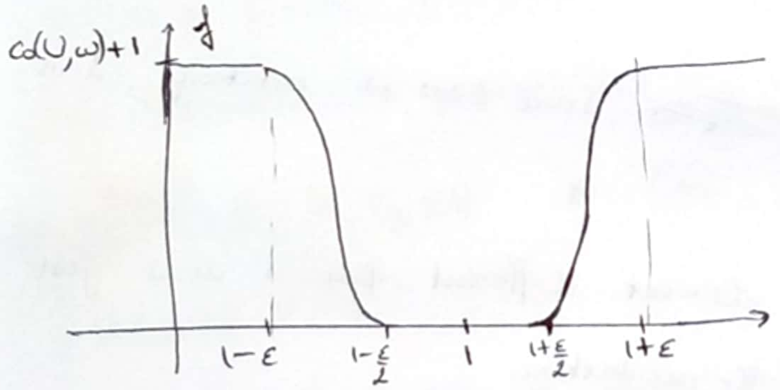
Proof: we are going to construct a special Hamiltonian on U that belongs to the set $\mathcal{H}(U,\omega)$ of functions used to define c_0.

~~Let S_λ be a surface in the family~~

~~Suppose~~ If $I = \{\lambda \mid 1-\rho < \lambda < 1+\rho\}$ for some $\rho > 0$, choose $0 < \varepsilon < \rho$.

Choose a smooth functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$
\begin{cases}
f(s) = c_0(U,\omega) + 1 & \text{for } s \le 1-\varepsilon,\ s \ge 1+\varepsilon \\
f(s) = 0 & \text{for } 1-\frac{\varepsilon}{2} \le s \le 1+\frac{\varepsilon}{2} \\
f'(s) < 0 & \text{for } 1-\varepsilon < s < 1-\frac{\varepsilon}{2} \\
f'(s) > 0 & \text{for } 1+\frac{\varepsilon}{2} < s < 1+\varepsilon
\end{cases}
$$



Define $F : U \to \mathbb{R}$, $x \mapsto f(H(x))$. $\Rightarrow$ F is constant on each S_λ and $F \in \mathcal{H}(U,\omega)$.

Note that the oscillation $m(F) = \max(F) - \min(F) = c_0(U,\omega) + 1 > c_0(U,\omega)$.

By definition of c_0, there exists a nonconstant periodic orbit $x(t)$ having period $0 < T \le 1$ of the system $\dot{x} = X_F(x)$, $x \in U$.

It is easy to show that $X_F(x) = f'(H(x))\, X_H(x)$. (*)

Moreover, $H(x(t))$ is constant in t. Indeed,

$$\frac{d}{dt} H(x(t)) = dH\big(X_F(x(t))\big) = -\omega\big(X_H(x(t)), X_F(x(t))\big) = 0.$$

$\rightsquigarrow$ $H(x(t)) \equiv \lambda$.

Since $x(t)$ is nonconstant, we have (in view of (*))

$$1-\varepsilon < \lambda < 1 - \frac{\varepsilon}{2} \quad \text{or} \quad 1+\frac{\varepsilon}{2} < \lambda < 1+\varepsilon.$$

If $f'(\lambda) = \tau \neq 0$, define $y : \mathbb{R} \to S_\lambda$, $y(t) = x\left(\frac{t}{\tau}\right)$.

y has period τT and satisfies $\dot{y} = X_H(y)$.

$\rightsquigarrow$ y is periodic orbit of X_H on S_λ.

By construction, $|\lambda - 1| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, λ is arbitrarily close to 1.

To get the statement for any other element different from 1 in I just replace 1 with element and repeat the construction.

~~Question we have~~ <u>Question</u> We have found solutions on a dense set of S_λ's. Does there exist a solution on $S_1 = S$?

If we know that the periods T_j of the orbits x_j on S_{λ_j} for $\lambda_j \to 1$ are uniformly bounded, then the answer is positive

Let us make this more precise.

Let g be a Riemannian metric on M.
$\langle , \rangle = g$

If $x(t)$ is a period solution, we define its length

$$\ell(x) = \int_0^T \|\dot{x}(t)\|\, dt$$

Possibly after shrinking U, we can assume $\frac{1}{C} \leq \|X_H\| \leq C$ on U.

for some $C > 0$.

$$\Rightarrow \frac{T_j}{C} \leq \ell(x_j) \leq C T_j \quad \forall j.$$

Proposition. Let $\lambda_j \to 1$ and assume (T_j) is bounded. Then $S = S_1$ admits a periodic solution.

Proof: normalize the periods to 1 by defining $y_j(t) = x_j(T_j t)$, $t \in [0,1]$.

$$\Rightarrow \begin{cases} \dot{y}_j(t) = T_j X_H(y_j(t)). & (*) \\ H(y_j(t)) \equiv \lambda_j. \end{cases}$$

Note that $T_j X_H(y_j(t))$ is bounded by assumption.

$\Rightarrow (y_j)$ is equicts and bounded. By the Arzelà-Ascoli theorem, we can assume $y_j \xrightarrow{C^0} y$. By using $(*)$, we see that this convergence is actually in the C^∞-topology. $\leadsto$ we get a 1-periodic solution y, with $H(y(t)) = 1$.

i.e. $\dot{y}(t) = X_H(y(t))$.

If the period T of y is not 0, we are done.

Suppose $T = 0$. $\Rightarrow$ $y_j \to y^*$, where $y^* \in S$ is a point.

$\Rightarrow X_H(y_j(t)) \to X_H(y^*). =: V$.

Since S is a regular level set, $X_H \neq 0$ on $S \Rightarrow V \neq 0$.

Note that $\langle X_H(y_j(t)), V\rangle \geq (1-\varepsilon)\|V\|^2$ for large j and $\varepsilon > 0$ small.

$\Rightarrow \frac{1}{T_j}\langle \dot{y}_j(t), V\rangle \geq (1-\varepsilon)\|V\|^2$ (this makes sense in local coordinates and we are using the standard euclidean product)

However, $0 = \int_0^{T_j} \frac{1}{T_j}\langle \dot{y}_j(t), V\rangle dt = \int_0^{T_j} \frac{d}{dt}\left(\frac{1}{T_j}\langle y_j(t), V\rangle\right) dt \geq T_j(1-\varepsilon)\|V\|^2$

$\Rightarrow \|V\| = 0$, contradiction ■

Remark We can apply Hofer-Zehnder to cpt hypersurfaces in $(\mathbb{R}^{2n}, \omega_0)$. Since we can always embed such surfaces in large enough balls, which have finite capacity, we can apply the theorem.

We are now going to restrict the class of hypersurfaces we consider in order to be able to apply the previous proposition

~~Assume S is the boundary of some~~ We are going to consider two classes (actually, the second is a subclass of the first).

(I): Let $S \subseteq (M, \omega)$ be a cpt hypersurface and assume S is the boundary of some compact symplectic mfd $(B, \omega) \subseteq (M, \omega)$.

Let (S_ε) be a parametrized family of surfaces modeled on S.

$\rightsquigarrow$ S_ε bounds a mfd B_ε. Assume the parametrization is such that $\varepsilon < \varepsilon' \Rightarrow B_\varepsilon \subseteq B_{\varepsilon'}$.

Monotonicity of c $\Longrightarrow$ $c(B_\varepsilon, \omega) \le c(B_{\varepsilon'}, \omega)$ if $\varepsilon \le \varepsilon'$.

Denote $C(\varepsilon) = c(B_\varepsilon, \omega)$, so that $\varepsilon \mapsto C(\varepsilon)$ is monotone increasing.

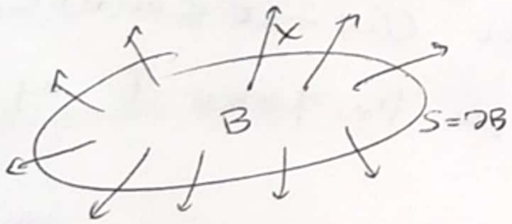
<u>Def.</u> S_{ε^*} is called of c-Lipschitz-type if there exist $L, \mu > 0$ such that

$$C(\varepsilon) \le C(\varepsilon^*) + L(\varepsilon - \varepsilon^*) \quad \forall\, \varepsilon^* \le \varepsilon \le \varepsilon^* + \mu.$$

<u>Exercise</u>. Show that this notion does depend on the choice of family modeled on S_{ε^*}.

<u>Example</u>. Suppose in a nbd of $S = \partial B$ there exists a Liouville vector field, i.e. a vector field X satisfying

$$\begin{cases} L_X \omega = \omega \\ X \pitchfork S. \end{cases}$$



Then, S is of c-Lipschitz type.

<u>Exercise</u> Prove this.

<u>Thm</u> Assume $c(M, \omega) < \infty$. If the cpt hypersurface $S \subseteq (M, \omega)$ bounds a symplectic mfd and is of c-Lipschitz type, then $P(S) \neq \emptyset$.

<u>Proof.</u> by assumption, $\exists$ (S_ε) family modeled on $S = S_0$ with

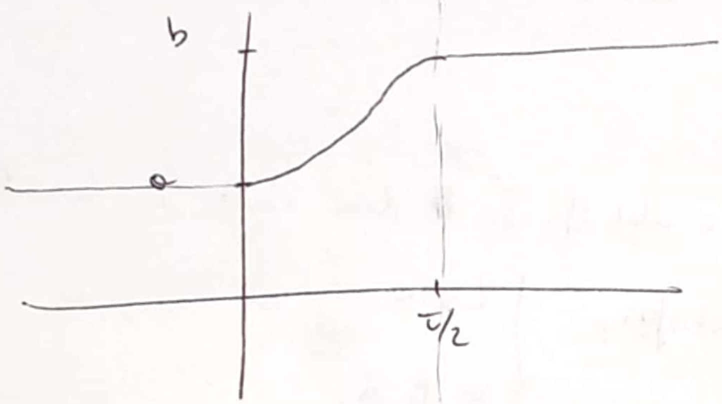
$$C(\varepsilon) \le C(0) + L\varepsilon \quad \forall\, 0 \le \varepsilon \le \mu.$$

Define the set $\mathcal{F}_\tau$ of functions $f: \mathbb{R} \to (C(0) - \tau L, \infty)$ for $0 < \tau < \mu$. with the following restrictions:

$$\begin{cases} f(s) = a & \text{if } s \leq 0 \\ f(s) = b & \text{if } s \geq \frac{\tau}{2} \\ 0 < f'(s) \leq c & \text{if } 0 < s < \frac{\tau}{2} \end{cases}$$

with $\begin{cases} C(0) - L\tau \leq a \leq C(0) \\ C(0) + 2L\tau \leq b \leq C(0) + 3L\tau \\ c \text{ large enough. but independent of } \tau. \end{cases}$

(we are going to be more specific at the end of the proof).



$f \in \mathcal{F}_\tau$.

Then $\mathcal{F}_\tau \neq \emptyset$. By definition of $c_0(B_0)$ there exists an admissible function $H \in \mathcal{H}_a(B_0, \omega)$ with oscillation $C(0) - L\tau \leq m(H) < C(0)$.

Choose $f \in \mathcal{F}_\tau$ with $a = m(H)$ and define the function F by

$$\begin{cases} F(x) = H(x) & \text{if } x \in B_0 \\ F(x) = f(\varepsilon) & \text{if } x \in S_\varepsilon, \quad 0 \leq \varepsilon \leq \tau. \\ F(x) = b & \text{if } x \notin \overline{B_\tau}. \end{cases}$$

Then, $F \in \mathcal{H}(B_\tau, \omega)$ and $m(F) = b \geq C(0) + 2L\tau > C(0) + L\tau \geq C(\tau)$

C_0-Lipschitz condition.

By definition, $\exists$ nonconstant periodic orbit $x(t)$ of X_F with period $0 < T \leq 1$ contained in B_τ.

Note that B_0 is invariant under the flow of X_F. Since $F|_{B_0} = H$ is admissible, we see that $x(t) \in B_\tau \setminus \overline{B_0}$ $\forall t$.

X.

$\Rightarrow \exists\, \varepsilon \in (0, \frac{\pi}{2})$ such that $x(t) \in S_\varepsilon \ \forall t$.

This argument works for every $0<\tau<\mu$.

By choosing a sequence $\tau_j \longrightarrow 0$, we get sequences F_j, ε_j and periodic orbits $x_j(t)$ satisfying

$$\begin{cases} \dot{x}_j = X_{F_j}(x_j). \\ x_j(t) \in S_{\varepsilon_j}, \quad \varepsilon_j \longrightarrow 0. \\ 0 < T_j \leq 1. \end{cases}$$

Now, consider the nbd U of S foliated by (S_ε). Define a function K on U by $K(x) = \varepsilon$ if $x \in S_\varepsilon$.

Note that $F_j(x) = f_j(K(x))$ $\forall x \in S_\varepsilon$, $0 \leq \varepsilon \leq \tau_j$.

In particular, for those points x, we get $X_{F_j}(x) = f_j'(K(x))\, X_K(x)$.

$\rightsquigarrow$ the periodic orbits x_j satisfy $\begin{cases} \dot{x}_j(t) = f'(\varepsilon_j) X_K(x_j(t)) \\ x_j(0) = x_j(T_j) \end{cases}$, $0 < T_j \leq 1$.

Reparametrize: $y_j(t) := x_j\left(\frac{t}{f'(\varepsilon_j)}\right)$

$$\rightsquigarrow \begin{cases} \dot{y}_j(t) = X_K(y_j(t)) \\ K(y_j(t)) = \varepsilon_j \end{cases}$$

The periods of the y_j's are given by $T_j f'(\varepsilon_j)$. Choose c large enough by independent of τ, for instance $c = 10L$ $\Rightarrow$ the periods of the y_j's are uniformly bounded. $\Rightarrow P(S) \neq \emptyset$.

Remark

Suppose S is a cpt, connected hypersurface in $(\mathbb{R}^{2n}, \omega_0)$. Then S is orientable (there are a few ways of seeing this, I might put one in the exercises) and it separates $\mathbb{R}^{2n}$ into two components, one bounded and the other one unbounded. The bounded component has finite capacity since we can embed it in some big enough ball. If S admits a parametrized family and is of co-Lipschitz type, we can apply the previous theorem.

(II) Hypersurfaces of contact type

This is another property of surfaces that ensures the existence of closed characteristics.

Let $S \subseteq (M, \omega)$ be a cpt hypersurface.

Def. We say that S is of contact type if $\exists$ Liouville vector field X defined in a nbd of S, i.e. a vector field satisfying

$$\begin{cases} \mathcal{L}_X \omega = \omega \\ X \pitchfork S. \end{cases}$$

Brief detour into contact geometry

Let M^{2n+1} be an orientable $(2n+1)$-dimensional mfd.

Def. A contact structure on M^{2n+1} is a cooriented hyperplane distribution $\xi \subset TM$ that is maximally nowhere integrable. This means that if $\alpha \in \Omega^1(M)$ is such that $\ker\alpha = \xi$ (this is possible by coorientability), then $\alpha \wedge (d\alpha)^n$ is a volume form.

Thus means that there does not exist any open set on which ξ can be integrated.

Examples (i) $\left(\mathbb{R}^{2n+1}, \alpha = dz + \sum_{i=1}^{n} x_i dy_i\right)$.

(ii) $S^{2n+1} \subseteq (\mathbb{R}^{2n+2}, \omega_0)$ with the contact structure $\ker \omega_0|_{S^{2n+1}}$.

Exercise Show that if $S \subseteq (M, \omega)$ is of contact type, then the form $\alpha := (i_X \omega)|_S$ is a contact form on S.

Exercise Prove that ~~star-shaped~~ strictly convex hypersurfaces are of contact type.

The contact type condition is nice because it gives a special parametrized family modeled on S. Indeed, the flow φ^t of X is defined for $|t| < \varepsilon$ (ε small enough, by compactness of S) and it defines a diffeo $\varphi^t : S \times (-\varepsilon, \varepsilon) \longrightarrow U$ onto a nbd of S.

Since $\mathcal{L}_X \omega = \omega$, we deduce that $\varphi^{t*}\omega = e^t \omega$.

Using this, it's easy to see that ~~$d\varphi^t : TS \to TS_t$ restricts to a~~ $d\varphi^t : \mathcal{L}_S \longrightarrow \mathcal{L}_{S_t}$ is a bundle isomorphism.

Thus means that φ^t induces a bijection $P(S) \longrightarrow P(S_t)$

We can extrapolate a definition out of this.

Def. A cpt hypersurface $S \subseteq (M,\omega)$ is called stable if there exists a parametrized family modeled on S having the property that the associated diffeo $\psi : S \times I \to U$ induces bundle isomorphisms

$$d\psi_\varepsilon : \mathcal{L}_S \longrightarrow \mathcal{L}_{S_\varepsilon}.$$

We can thus rephrase the existence theorem for closed characteristics as follows:

Thm. Assume $S \subseteq (M,\omega)$ admits a nbd U with $c_0(U,\omega) < \infty$. If S is stable, then $P(S) \neq \emptyset$.

Example A stable surface need not be of contact type. Consider a closed symplectic mfd (N,ω_1) and

$$M = (N \times \mathbb{R}^2, \underbrace{\omega_1 \oplus \omega_0}_{\omega}).$$

Let $S := \{(x,v) \mid \|v\| = 1\} \subseteq M$ be a cpt hypersurface and define the parametrization $\psi_\varepsilon(x,v) = (x, \varepsilon v)$.

$\rightsquigarrow S_\varepsilon = \{(x,v) \mid \|v\| = \varepsilon\}$.

Clearly, S is stable. However, we claim it is not of contact type.

If it were, we would be able to find a 1-form α on S such that $d\alpha = j^*\omega$, where $j : S \hookrightarrow M$ is the inclusion. Let $i : N \hookrightarrow N \times \{(1,0)\}$ be the inclusion. Then we have

$$\underset{\substack{\| \\ d(i^*\alpha)}}{i^* d\alpha} = i^* j^* \omega = (ji)^*\omega = \omega_1 \implies \omega_1 \text{ is exact, contradiction (as } N \text{ is closed).}$$

XIV