Symplectic Geometry 4: symplectic reduction

Symplectic reduction (Marsden-Weinstein-Meyer) is a way to produce new symplectic manifolds as certain quotients.
For Hamiltonian systems with symmetries it underlies the reduction of the system, X_H, to a reduced system $X_{\overline{H}}$ on a lower dimensional space.

So far, as examples of symplectic manifolds we have:

- co-tangent bundles T^*Q, $\omega = d\lambda$ (λ canonical 1-form).
- oriented surfaces Σ (ω an oriented area form).

And, as non-examples we have observed that

- S^{2n}, $n > 1$ does not admit a symplectic structure.

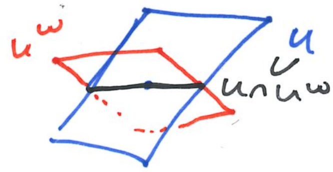
Some 'new' examples we will obtain by symplectic red. are:

- $\mathbb{C}P^n$, ω_{FS}
- co-adjoint orbits: $\mathcal{O}_\mu = \{ Ad_g^* \mu : g \in G \} \subset \mathfrak{g}^*$, ω_μ.

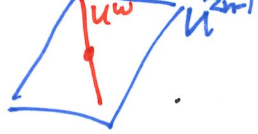
Recall that in linear symplectic geometry, we have for $U \subset (V, \omega)$ a subspace then

$\overline{U} = U/(U \cap U^\omega)$ has an induced symplectic form $\overline{\omega}$

[note $U \cap U^\omega = \ker \omega|_U$].



In particular, if $U \subset V$ is a hyperplane ($\dim U = 2n-1 = \dim V - 1$) then $l_U = U^\omega \subset U$ is a line in V contained in U

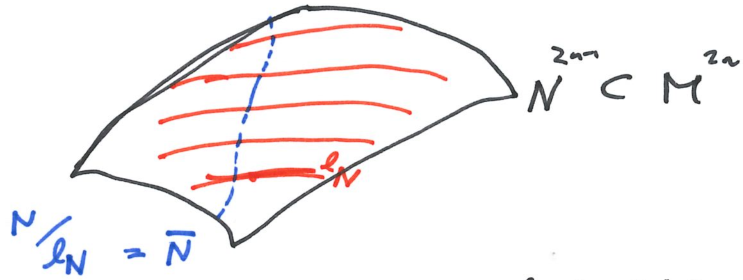


For manifolds we have the following analogues:

① Let $N \subset M$ a regular hypersurface ($\dim N = \dim M - 1 = 2n-1$) with characteristic line field $l_N = (TN)^\omega \subset TN$. Then, if a manifold the quotient

$$\overline{N} = N/l_N = N/\{x \sim y \text{ if } x, y \text{ are on a common integral curve of } l_N\}$$

is a symplectic manifold with an induced symplectic structure $\overline{\omega}$.

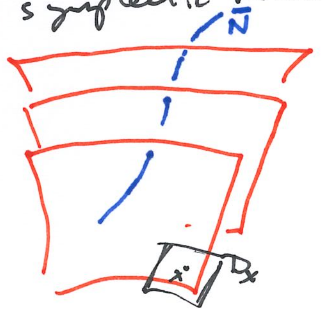


② more generally, we will see if $N \subset M$ is a submanifold and the distribution $D_x = T_xN \cap (T_xN)^\omega \subset T_xN$ on N has constant rank ($\dim D_x = k = cst.\ \forall x \in N$)

[NOTE $D_x = \ker\{\omega_x|_N\}$] then D is an integrable distribution on N [ie D are the tangent spaces to a k-dim. foliation of N] and if a manifold, the 'leaf space':

$$\overline{N} = N/D = N/\{x \sim y \text{ if } x \text{ and } y \text{ lie in a connected integral } k\text{-surface of } D\}$$

is then a symplectic manifold with an induced symplectic structure $\overline{\omega}$.



[* recall: given a distribution of k-planes: $D_x \subset T_xN$ an integral submanifold is $I \subset N$ s.t. $T_xI \subset D_x$. The distribution is integrable if there exist dimension $\dim(D_x) = k = cst.$ integrable submanifolds through each pt. By Frobenius theorem, this is the case when (diff) vector flds tangent to D are closed under Lie bracket.]

The last two claims are special cases of:

$\boxed{\text{Prop:}}$ Let N a mfld. with a closed 2-form $\beta \in \Omega^2(N)$ such that

$$D := \ker\beta = \{X : \iota_X \beta = 0\}$$

has constant rank ($\dim D_x = k = cst.$). [Note: we call β a 'pre-symplectic' structure on N]. Then

1) D is an integrable distribution on N

2) if the quotient $\bar{N} := N/D$ is a manifold, then it has an induced symplectic structure $\bar\omega$ through:

$$\pi^* \bar\omega = \beta \qquad (\pi : N \to \bar N).$$

prf: 1) we will check involutivity ($[D,D] = D$) of D (Frobenius thm.) Let X, Y be vector fields on N tangent to D. Then:

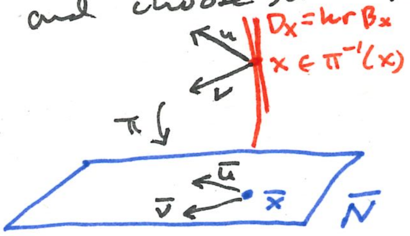
$$(*) \quad L_X \beta = \iota_X \cancel{d\beta} + d(\cancel{\iota_X \beta}) = 0$$

β closed $\qquad X \in D = \ker\beta$

and $\quad 0 = L_X(0) = L_X(\iota_Y \beta) = \iota_{[X,Y]}\beta + \iota_Y \cancel{L_X \beta}^{\,0} = \iota_{[X,Y]}\beta$

so that $[X,Y] \in D$ is tangent to D and D is integrable.

2) We check $\bar\omega$ is well-defined. Let $\bar x \in \bar N$, $\bar u, \bar v \in T_{\bar x}\bar N$ and choose some $x \in \pi^{-1}(\bar x)$, $u, v \in T_x N$ s.t. $\pi_* u = \bar u$, $\pi_* v = \bar v$:



Then $\pi^* \bar\omega = \beta$ reads:

$$\bar\omega_{\bar x}(\bar u, \bar v) = \beta_x(u, v)$$

and we want to show is independent of choice of x and lifts u, v.

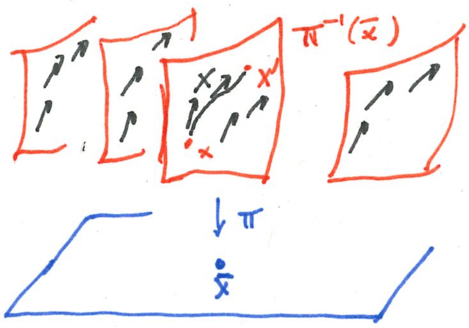
1^{st} keeping x fixed, if $\tilde u, \tilde v \in T_x N$ have

$$\pi_* \tilde u = \bar u = \pi_* u \qquad \pi_* \tilde v = \bar v = \pi_* v,$$

then $\pi_*(\tilde u - u) = 0$, ie $\tilde u - u \in T_x(\pi^{-1}(\bar x)) = \ker\beta_x$ [likewise for $\tilde v, v$]

so that $\beta_x(\tilde u, \tilde v) = \beta_x(u, v)$.

2^{nd} let $x' \in \pi^{-1}(\bar{x})$ some other preimage on the leaf, and choose some (local) vector field $X \in D$ whose flow φ_t has $\varphi_1(x) = x'$



set $\varphi := \varphi_1$, and

$u' = \varphi_* u$, $v' = \varphi_* v \in T_{x'}N$

Then, since $X \in D$ is tangent to the fibers $\pi^{-1}(\bar{x})$, we have:

$$\pi = \pi \circ \varphi_t = \pi \circ \varphi$$

in particular: $\pi_* u' = \pi_* \varphi_* u = \pi_* u = \bar{u}$ (and $\pi_* v' = \pi_* v = \bar{v}$), and by $(*)$, $L_X \beta = 0$, so that $\varphi^* \beta = \beta$ we have:

$$\beta_x(u,v) = (\varphi^* \beta)_x(u,v) = \beta_{x'}(u',v').$$ So that $\bar{\omega}$ is well-def'd.

It is non-degenerate since β on $TN/\ker\beta$ is non-degenerate, and closed since π_* is onto, so π^* is injective and

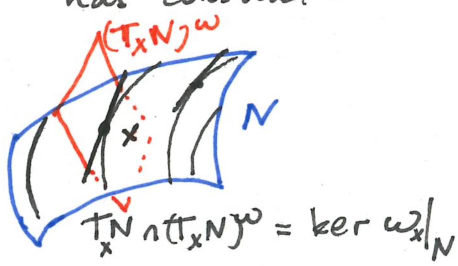
$$0 = \pi^* d\bar{\omega} = d\beta.$$

$\square$

<u>Remark:</u> As special cases of this last proposition, suppose

$N \subset (M, \omega)$ is a submanifold such that

$D_N = (TN)^\omega \cap TN = \ker \omega|_N \subset TN$ has constant rank.



Then, if a manifold, the quotient

$\bar{N} = N/D_N$ is a symplectic manifold,

and its symplectic structure $\bar{\omega}$ is determined through:

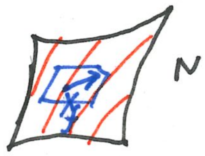
$$\begin{array}{ccc} N & \overset{\iota}{\hookrightarrow} & M \\ {\scriptstyle \pi}\downarrow & & \\ \bar{N} & & \end{array} \qquad \pi^* \bar{\omega} = \iota^* \omega \; (= \omega|_N).$$

In particular, for co-isotropic submanifolds, we have the following structure for computing 'upstairs' on M vs. 'downstairs' on $\bar{N}$:

$\boxed{\text{Prop:}}$ Let $N \subset M$ be a co-isotropic submanifold where $(TN)^\omega \subset TN$ has constant rank, and suppose the quotient $\bar{N} = N/(TN)^\omega$ is a manifold with induced symplectic structure $\bar{\omega}$ and submersion $N \xrightarrow{\pi} \bar{N}$. Then:

1) if $f: M \to \mathbb{R}$ has $f|_N = \pi^* \bar{f} = \bar{f} \circ \pi$ for some $\bar{f}: \bar{N} \to \mathbb{R}$

then $X_f(n) \in T_n N \ \forall n \in N$



2) for $f, \bar{f}$ as in (1) then

$$\pi_* X_f = X_{\bar{f}}$$

where $X_{\bar{f}}$ is the Hamiltonian v.f. of $\bar{f}: \bar{N} \to \mathbb{R}$ with respect to $\bar{\omega}$.

3) for $f, g: M \to \mathbb{R}$ s.t. $f|_N = \pi^* \bar{f}$, $g|_N = \pi^* \bar{g}$ for $\bar{f}, \bar{g}: \bar{N} \to \mathbb{R}$,

then $$\{f, g\}|_N = \pi^* \{\bar{f}, \bar{g}\}.$$

prf: 1) for $n \in N$ and $v \in (T_n N)^\omega \subset T_n N$ we have:

$$\omega_n(v, X_f(n)) = d_n f(v) = d_{\bar{n}} \bar{f} \cdot d_n \pi(v) = 0$$

(since $(T_n N)^\omega = \ker d_n \pi$). This holding for all $v \in (T_n N)^\omega$, we have $X_f(n) \in ((T_n N)^\omega)^\omega = T_n N$.

2) for $v \in T_n N$ we have (set $\bar{v} = \pi_* v$):

$$d_n f(v) = \omega_n(v, X_f(n)) = \bar{\omega}_{\bar{n}}(\bar{v}, \pi_* X_f(n))$$

$$\parallel$$

$$d_{\bar{n}} \bar{f}(\bar{v}) = \bar{\omega}_{\bar{n}}(\bar{v}, X_{\bar{f}}(\bar{n})), \text{ so that } \pi_* X_f = X_{\bar{f}}.$$

3) $\{f,g\}(n) = \omega_n(X_g, X_f) = \bar{\omega}_{\bar{n}}(\pi_* X_g, \pi_* X_f)$

$= \{\bar{f}, \bar{g}\}(\bar{n})$ (since $\pi_* X_g = X_{\bar{g}}$, $\pi_* X_f = X_{\bar{f}}$ by (2)).

□

$\boxed{\text{Example:}}$ Consider our standard symplectic vector space

$$\mathbb{R}^{2n+2},\ dp\wedge dq \longleftrightarrow \mathbb{C}^{n+1},\ \mathrm{Im}\langle\cdot,\cdot\rangle$$

we have the hypersurface of the standard sphere:

$$S^{2n+1} \subset \mathbb{C}^{n+1}$$

and so a symplectic quotient $S^{2n+1}/\ell_{S^{2n+1}}$, where the 'characteristic line field' $\ell_{S^{2n+1}} \subset TS^{2n+1}$ is spanned by

$$X_H = p\cdot\partial_q - q\cdot\partial_p \longleftrightarrow X_H(z) = iz \quad (z\in S^{2n+1}),$$

[by writing $S^{2n+1} = \{ \underbrace{\frac{q_1^2+p_1^2}{2}+\dots+\frac{q_{n+1}^2+p_{n+1}^2}{2}}_{H} = \frac{1}{2} \}$ so that $\ell_{S^{2n+1}} = (TS^{2n+1})^\omega$ is spanned by X_H]. The integral curves of $\ell_{S^{2n+1}}$ are the orbits of the circle action $S^1 \circlearrowright S^{2n+1}$:

$$\{ e^{i\vartheta}\cdot z = (e^{i\vartheta}z_1, \dots, e^{i\vartheta}z_{n+1}) : \vartheta \in \mathbb{R}/2\pi\mathbb{Z} \}$$

and so the symplectic quotient is:

$$S^{2n+1}/\ell_{S^{2n+1}} = S^{2n+1}/S^1 \approx \mathbb{C}P^n.$$

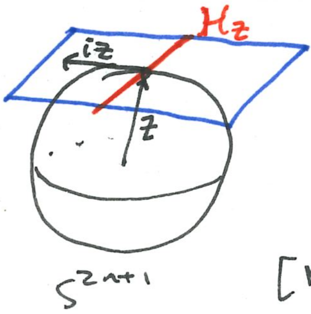
Hence $\mathbb{C}P^n$ has an induced symplectic structure we denote

$$\omega_{FS}$$

(for 'FUBINI-STUDY'). It can be described more explicitly as follows:

let $z \in S^{2n+1} \xrightarrow{\pi} [z] \in \mathbb{C}P^n$

and set $\mathcal{H}_z := \{z, iz\}^\perp \subset T_z S^{2n+1}$



since iz spans $\ker d_z\pi$, we have that

$$d\pi|_{\mathcal{H}_z} : \mathcal{H}_z \xrightarrow{\sim} T_{[z]}\mathbb{C}P^n \quad \text{is an isomorphism.}$$

[NOTE that $\mathcal{H}_z$ is a complex vector subspace of $\mathbb{C}^n$]

through $d\pi|_{\mathcal{H}_z}$ we can induce structures on $\mathbb{C}P^n$ by $e^{i\theta}\cdot z$ invariant structures on $\mathbb{C}^n \supset \mathcal{H}_z$:

for $\bar{u}, \bar{v} \in T_{[z]}\mathbb{C}P^n$, let $z \in S^{2n+1}$, $u, v \in \mathcal{H}_z \subset T_z S^{2n+1}$ with $d\pi(u) = \bar{u}$, $d\pi(v) = \bar{v}$ then:

$$\omega_{FS}(\bar{u},\bar{v}) = \operatorname{Im}\langle u,v\rangle \left(= \omega(u,v) = u\cdot iv\right)$$

is the reduced symplectic str. on $\mathbb{C}P^n$. Likewise:

$$g_{FS}(\bar{u},\bar{v}) = \operatorname{Re}\langle u,v\rangle \ (= u\cdot v)$$

is the induced FUBINI-STUDY metric on $\mathbb{C}P^n$, and we have a complex structure on $\mathbb{C}P^n$ by

$$i\bar{u} = d\pi|_{\mathcal{H}_z}(iu).$$

so that $\omega_{FS}(\bar{u},\bar{v}) = g_{FS}(\bar{u}, i\bar{v})$.

Remark: Lets recall that we call an almost complex structure J on a manifold M a complex vector bundle structure on $TM \to M$, ie. a cplx. str. $J_x : T_xM \circlearrowleft$ $J_x^2 = -Id$ on each tagent space. Whereas we call a complex structure on a manifold M (or say M is a complex manifold) if M admits an atlas of $\mathbb{C}^n$-valued charts with holomorphic transition functions.

Any complex manifold has also an almost complex structure which corresponds in the above $\mathbb{C}^n$-valued charts to multiplication by i. (exercise: check multiplication by 'i' in a cplx. atlas as above yields an almost cplx str. on M).

The almost complex structures J on M which correspond to 'multiplication by i' in such a cplx atlas are called the 'integrable almost cplx structures' on M are also just the complex structures on M. There is an analytic condition for when an almost complex structure J on M underlies multiplication by 'i' in some cplx atlas (cplx str.) on M which is the Nijenhuis tensor:

$$N_J(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

then almost cplx str. J is an integrable almost cplx str. (ie underlies multiplication by 'i' in some $\mathbb{C}^n$-valued atlas on M) iff $N_J \equiv 0$ (this is a theorem). So complex structures are very special types of almost complex structures (and we recall on S^6 there is a known almost cplx structure — which is not complex — but it is an open question if there is any cplx. structure on S^6).

$\boxed{\text{Remark:}}$ The symplectic manifolds (M,ω) which admit ω-compatible almost complex structures J that are integrable (so are associated to a complex structure on M) are then rather special. They are called (positive) Kähler manifolds:

$\boxed{\text{Def:}}$ Let (W,i) be a complex manifold. If there is a symplectic structure ω on W s.t. $i \in \mathcal{J}(W,\omega)$ then we call (W,i,ω) a (positive) Kähler manifold.

Note that a positive Kähler manifold then has an associated Rimanian metric $g(u,v) = \omega(iu,v)$.

$\boxed{\text{Prop:}}$ For a symplectic manifold (M,ω) let $J \in \mathcal{J}(M,\omega)$ be an ω-compatible almost complex structure. Then if $N \subset M$ is a submanifold with:

$$JTN = TN,$$

we have N is a symplectic manifold with the restriction $\omega|_N$ of ω to N.

prf: Set $\bar\omega = \omega|_N$. Then $d\bar\omega = 0$ is closed still, so we need to check it is non-degenerate. Note that, by def. of ω-compatible, we have a positive def. Riemannian metric

$$g(u,v) = \omega(Ju, v) \quad \text{on } M \; [\text{or } \omega(u,v) = g(u, Jv)]$$

Now let $x \in N$ and $u \in T_xN$ and suppose

$$0 = \omega_x(u,v) = g_x(u, Jv) \quad \forall v \in T_xN.$$

then, since $JTN = TN$ and J is invertible, we have

$$0 = g_x(u, v') \quad \forall v' \in T_xN \quad (v' = Jv \in T_xN).$$

i.e. $u = 0$ since restriction of a Rmn. metric to N is still an non-degen. Rmn. metric on N. $\square$

Remark: As a corollary of this last proposition, we have that in any (positive) Kähler manifold that any complex submanifold is a symplectic manifold (a symplectic submfld.) with the restriction of the symplectic form. In particular, returning to $(\mathbb{C}P^n, \omega_{FS})$ any complex (even immersed) submanifold $N \subset \mathbb{C}P^n$ is then a symplectic manifold with $\omega_{FS}|_N$. For example any (non-singular) algebraic variety:

$$N = \mathbb{P}\{Z_0^3 = Z_1^3 + Z_2^2 Z_3\} \subset \mathbb{C}P^3 \ni [Z_0 : Z_1 : Z_2 : Z_3].$$

Exercise: Check that for the standard cplx structure on $\mathbb{C}P^n$ induced by the affine coordinates:

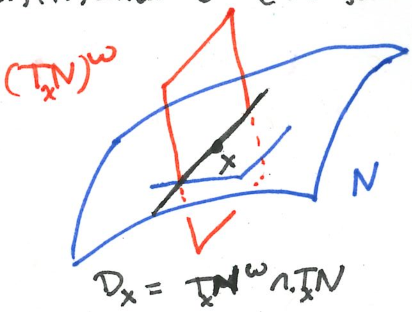
$$\mathbb{C}P^n \supset \{Z_0 \neq 0\} \longleftrightarrow \mathbb{C}^n$$

$$[Z_0 : Z_1 : \dots : Z_n] \longleftrightarrow \left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right) = (z_1, \dots, z_n)$$

that multiplication by i on $\mathbb{C}^n$ induces an ω_{FS}-compatible (integrable) almost cplx. structure on $\mathbb{C}P^n$.

Moment maps

Our underlying proposition on pg. 3 is not in general very useful in practice: to determine the quotient $\bar{N} = N/_{D}$ requires finding the integral submanifolds of the (integrable) distribution D (ie solving some system of PDE's)



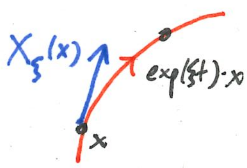
We will find certain conditions for group actions in which one can describe these quotient spaces more explicitly (in particular, without needing to integrate the distribution D).

Def: Let a Lie group G act on (M,ω) : $G \circlearrowright M$. The action is a symplectic action if $\varphi_g^* \omega = \omega$ $\quad (\varphi_g(m) = g\cdot m)$.

The infinitesimal generators of this action are the vector fields:

$$X_\xi(m) = \frac{d}{dt}\Big|_0 \exp(\xi t)\cdot m = \frac{d}{dt}\Big|_0 \gamma(t)\cdot m \quad , \quad \xi \in \mathfrak{g}$$

(where $\gamma(t) \in G$ has $\gamma(0) = e$ $\quad \dot{\gamma}(0) = \xi \in T_e G = \mathfrak{g}$)



Note: since the action is symplectic and the flow of X_ξ is $\varphi_t(m) = \exp(\xi t)\cdot m$, we have

$$L_{X_\xi}\omega = 0 = d(\iota_{X_\xi}\omega)$$

so that $X_\xi \in sp(M,\omega) = \{X \in \mathfrak{X}(M) : L_X \omega = 0 = d(\iota_X \omega)\} \subset \mathfrak{X}(M)$.

are all symplectic vector fields.

As well, if we use left actions we compute that:

$$[X_\xi, X_\eta] = X_{[\eta,\xi]}$$

(Recall that $Ad_g : \mathfrak{g} \circlearrowleft$ $\xi \mapsto \frac{d}{dt}\big|_0 g\exp(\xi t)g^{-1}$

and $ad_\xi : \mathfrak{g} \circlearrowleft$ $\eta \mapsto \frac{d}{dt}\big|_0 Ad_{\exp(\xi t)}\eta$ has $ad_\xi(\eta) = [\xi,\eta]$

* for Matrix groups: $Ad_g \xi = g\xi g^{-1}$ and $ad_\xi(\eta) = \xi\eta - \eta\xi$ *)

Note that using right actions we have $[X_\xi, X_\eta] = X_{[\xi,\eta]}$.

Then a symplectic action $G \circlearrowright M$ has associated map:

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{sp}(M,\omega) \\ \xi &\longmapsto X_\xi \end{aligned}$$

where $\iota_{X_\xi}\omega$ is closed. As a 1st condition to have something more 'manageable' we can ask:

¿ when are the symplectic vector fields X_ξ all Hamiltonian vf's ?

ie when does a group action yield X_ξ's such that $\iota_{X_\xi}\omega$ are exact. We can look at the question with the diagrams (exact sequence):

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathfrak{sp}(M,\omega) \\ H &\longmapsto X_H \quad (\iota_{X_H}\omega = -dH) \end{aligned}$$

$$\begin{aligned} \mathfrak{sp}(M,\omega) &\longrightarrow H^1(M) \\ X &\longmapsto [\iota_X\omega] \end{aligned}$$

for M connected, the Hamiltonian functions giving vanishing symplectic gradients are the constants functions, so we have the exact sequence:

$$0 \to \underset{\substack{\rotatebox{90}{$\simeq$} \\ \mathbb{R}\,(M \text{ connected})}}{H^0(M)} \to C^\infty(M) \to \mathfrak{sp}(M,\omega) \to H^1(M) \to 0$$

and we are seeking a lift of:

$$\begin{tikzcd} C^\infty(M) \arrow[r] & \mathfrak{sp}(M,\omega) \\ & \mathfrak{g} \arrow[u] \arrow[ul, dashed, "?"] \end{tikzcd}$$

* note such a lift (if it exists) is then only defined up to addition of constants on the Hamiltonians *.

Because functions are easier to work with than closed 1-forms we define:

$\boxed{\text{Def}}$: A symplectic action $G \curvearrowright M$ is called a weakly Hamiltonian action if $\iota_{X_\xi}\omega$ is exact $\forall \xi \in \mathfrak{g}$.

$\boxed{\text{Prop}}$: If $G \curvearrowright M$ is a symplectic action and:

1) $H^1(M) = 0$, OR

2) $M, \omega = d\lambda$ is an exact symplectic mfld and $G \curvearrowright M$ is an exact symplectic action ($\varphi_g^*\lambda = \lambda$), OR

3) $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 0$

then $G \curvearrowright M$ is a weakly Hamiltonian action.

pf: 1) ✓ 2) $0 = L_{X_\xi}\lambda = \iota_{X_\xi}\omega + d(\iota_{X_\xi}\lambda)$ so $\iota_{X_\xi}\lambda$ is a Hamiltonian function for X_ξ.

3) for $[\xi,\eta] \in [\mathfrak{g},\mathfrak{g}]$, then:

$$L_{X_\xi}(\iota_{X_\eta}\omega) = \iota_{[X_\xi,X_\eta]}\omega + \iota_{X_\eta}\cancelto{0}{L_{X_\xi}\omega} = \iota_{X_{[\eta,\xi]}}\omega$$

and ;

$$\mathcal{L}_{X_\xi}(\iota_{X_\eta}\omega) = \iota_{X_\xi} d(\iota_{X_\eta}\omega) + d(\iota_{X_\xi}\iota_{X_\eta}\omega)$$

($\iota_{X_\eta}\omega$ is closed) $\rightarrow 0$

so that $\iota_{X_{[\eta,\xi]}}\omega = d(\iota_{X_\xi}\iota_{X_\eta}\omega)$ is exact. In particular if $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]=0$ then any $\varphi \in \mathfrak{g}$ is given by $\varphi = [\xi,\eta]$ some $\xi, \eta \in \mathfrak{g}$ and $X_\varphi = X_{[\xi,\eta]}$ is then a Hamiltonian v.f. $\square$.

So, for a weakly Ham. action $G \circlearrowright M$ we have not only the induced map $\mathfrak{g} \to \mathfrak{sp}(M,\omega)$, $\xi \mapsto X_\xi$ but also a map:

$$\begin{cases} \mathfrak{g} \longrightarrow C^\infty(M) \\ \xi \longmapsto \mu_\xi \end{cases} \qquad (\iota_{X_\xi}\omega = -d\mu_\xi)$$

And we can call the weak moment map

$$\begin{cases} \mu : M \longrightarrow \mathfrak{g}^* \\ \mu(m)(\xi) := \mu_\xi(m). \end{cases}$$

Exercise: Show that for a weakly Hamiltonian action $G \circlearrowright M$ one can always make some choice of Hamiltonians so that $\mathfrak{g} \to C^\infty(M)$ is a linear map (consider a basis $\xi_1, \dots, \xi_k$ of $\mathfrak{g}$ with Hamiltonians $\mu_j = \mu_{\xi_j}$ and show that $\mu_\xi := c^1\mu_1 + \dots + c^k\mu_k$ for $\xi = c^1\xi_1 + \dots + c^k\xi_k$ is a Hamiltonian for X_ξ).

Example: Consider a co-tangent bundle T^*Q, $\omega = d\lambda$. Then any diffeomorphism $f : Q \circlearrowleft$ has its co-tangent lift to a diffeo $\hat{f} : T^*Q \circlearrowleft$ $\quad [\hat{f}(q,p) = (f(q), p \circ d_q f^{-1})]$

Exercise: Show that $\hat{f}^* \lambda = \lambda$ is an exact symplectomorphism of T^*Q, $\omega = d\lambda$ (recall $\lambda_{(q,p)}(\xi) = p(\pi_* \xi)$) for $\pi : T^*Q \to Q$ & $\xi \in T_{(q,p)}(T^*Q)$.

In particular, any Lie group action

$$G \circlearrowright Q$$

we may take its co-tangent lift to have an exact symplectic action

$$G \circlearrowright T^*Q \qquad \varphi_g^* \lambda = \lambda .$$

which is in particular a (weakly) Hamiltonian action by (2) of the last prop. having Hamiltonians:

$$\left[\mu_\xi = \iota_{X_\xi} \lambda \right].$$

Example: Consider $SO_3 \circlearrowright \mathbb{R}^3 \ni q$ by $q \longmapsto A \cdot q$

its cotangent lift to $T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3 \ni (q,p)$ is by $(q,p) \longmapsto (Aq, Ap)$, and the generating vector fields of this action are $X_\xi(q,p) = (\xi q, \xi p)$ for $\xi \in \mathfrak{so}_3$.

if we identify $\quad \mathfrak{so}_3 \longleftrightarrow \mathbb{R}^3$ $\quad$ via 'cross product':

$$\xi \longleftrightarrow \vec{\xi} \qquad \xi(\vec{u}) = \vec{\xi} \times \vec{u}$$

then the (weak) moment map of this action is:

$$\mu(q,p)(\vec{\xi}) = p \cdot (\xi q) = \vec{\xi} \cdot (q \times p)$$

('angular momentum' of q, $p = \dot{q}$)

Among weakly Ham. actions we will be able to actually say something useful for:

Def: A weak Hamiltonian action $G \circlearrowright M$ is called an Hamiltonian action if it has an (equivariant) moment map

$$\mu : M \to \mathfrak{g}^* \qquad \mu(g\cdot m) = Ad^*_{g^{-1}}\mu(m), \forall g \in G, m \in M.$$

Prop: A weak Ham. action is Hamiltonian action iff:

$$\{\mu_\xi, \mu_\eta\} = \mu_{[\xi,\eta]}.$$

prf: let $m \in M$ and $\xi, \eta \in \mathfrak{g}$ then we differentiate

$$\mu(\exp(\xi t)\cdot m)(\eta) = (Ad^*_{\exp(-\xi t)}\mu(m))(\eta)$$

at $t=0$. For the left side:

$$\frac{d}{dt}\Big|_0 \mu(\exp(\xi t)\cdot m)(\eta) = \frac{d}{dt}\Big|_0 \mu_\eta(\exp(\xi t)\cdot m) = d\mu_\eta(X_\xi(m))$$

$$= \omega_m(X_\xi, X_\eta) = \{\mu_\eta, \mu_\xi\}(m).$$

For the right side:

$$\frac{d}{dt}\Big|_0 Ad^*_{\exp(-\xi t)}\mu(m)(\eta) = \frac{d}{dt}\Big|_0 \mu(m)(Ad_{\exp(-\xi t)}\eta)$$

$$= \mu(m)([\eta, \xi]) = \mu_{[\eta,\xi]}(m)$$

so that if μ is equivariant then $\{\mu_\xi, \mu_\eta\} \overset{(*)}{=} \mu_{[\xi,\eta]}$, and conversely if the above $(*)$ holds then (G being connected) we integrate the above steps: ($Ad_{\exp(-\xi t)}\mu(m) = \mu(\exp(\xi t)\cdot m) = $ cst.). $\square$

in particular:

Prop: If $G \curvearrowright M, \omega = d\lambda$ is an exact symplectic action ($\varphi_g^* \lambda = \lambda$) then it is a Hamiltonian action.

prf: Take $\mu_\xi := \iota_{X_\xi} \lambda$. Then for any $\eta \in \mathfrak{g}$:

$$d\mu_\xi(X_\eta) = L_{X_\eta}(\mu_\xi) = \iota_{[X_\eta, X_\xi]}\lambda = \iota_{X_{[\xi,\eta]}}\lambda = \mu_{[\xi,\eta]}$$

and, on the other hand: $d\mu_\xi(X_\eta) = \omega(X_\eta, X_\xi) = \{\mu_\xi, \mu_\eta\}$. $\square$

So, for example, a co-tangent lift of $G \curvearrowright Q$ to $G \curvearrowright T^*Q$ is always a Hamiltonian action.

Theorem: (Marsden-Weinstein-Meyer Symplectic Reduction)

Let $G \curvearrowright M$ a Hamiltonian action with G-equivariant moment map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

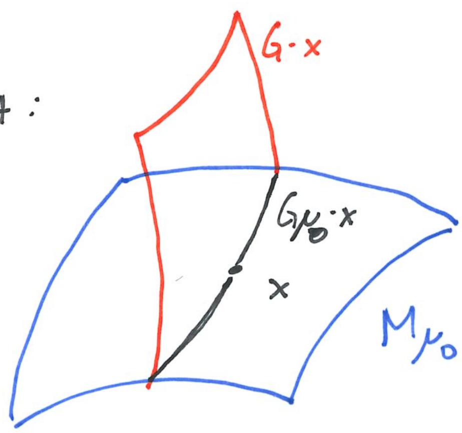
Set $M_{\mu_0} := \{\mu = \mu_0\}$ a regular lvl set of μ, and

$$G_{\mu_0} := \{ g \in G : Ad^*_{g^{-1}} \mu_0 = \mu_0 \} \subset G.$$

If $\overline{M}_{\mu_0} := M_{\mu_0}/G_{\mu_0}$ is a manifold, then it is a symplectic manifold, with induced symplectic structure $\overline{\omega}_{\mu_0}$ through:

$$\begin{tikzcd} M_{\mu_0} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ \overline{M}_{\mu_0} & \end{tikzcd} \qquad \pi^* \overline{\omega}_{\mu_0} = \iota^* \omega.$$

prf: let $x \in M_{\mu_0}$. we will show that:



$$(*) \quad (T_x M_{\mu_0})^\omega = T_x(G \cdot x)$$

so that, by G-equivariance of μ:

$$(T_x M_{\mu_0}) \cap (T_x M_{\mu_0})^\omega = T_x(G_{\mu_0} \cdot x)$$

and, in particular, then by the proposition on pg. 3, we have $\overline{M}_{\mu_0} = M_{\mu_0}/G_{\mu_0}$ with $\overline{\omega}_{\mu_0}$ as claimed.

So we just need to show $(*)$. Note that:

$$T_x(G\cdot x) = \{X_\xi(x) : \xi \in \mathfrak{g}\}$$

then for any $v \in T_x M_{\mu_0}$, say $v = \dot\gamma(0)$ for $t \mapsto \gamma(t) \in M_{\mu_0}$

we compute:

$$\omega_x(v, X_\xi(x)) = d_x\mu_\xi(v) = \frac{d}{dt}\Big|_0 \mu_\xi(\gamma(t))$$

$$= \frac{d}{dt}\Big|_0 \mu(\gamma(t))(\xi) = \frac{d}{dt}\Big|_0 \mu_0(\xi) = 0.$$

so that $T_x(G\cdot x) \subset (T_x M_{\mu_0})^\omega$. But since ($M_{\mu_0}$ is a regular lvl set)

$$\dim M_{\mu_0} = \dim M - \dim G = \dim(M) - \dim(G\cdot x),$$

we have equality so that $(*)$ holds and we are done. $\square$

$\boxed{\text{Examples:}}$ 1) $S^1 \circlearrowright \mathbb{C}^n$, $\omega = \mathrm{Im}\langle\cdot,\cdot\rangle$ has moment map

$$\mathbb{C}^n \to \mathbb{R}, \quad z \to \frac{|z|^2}{2}$$

the symplectic reductions are $\mathbb{CP}^n$, $c\cdot\omega_{FS}$ for c a constant.

2) Let X_H be a complete Hamiltonian vector field on M (its flow φ_t is defined for all time). The the $\mathbb{R}$-action

$$t\cdot x = \varphi_t(x) \quad \text{on } M$$

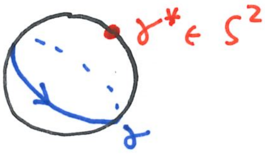
has moment map $H : M \to \mathbb{R}$. The symplectic reductions (if the quotient spaces are manifolds) are the 'manifolds of orbits':

$$\{H = c\} \big/ \{x \sim \varphi_t(x)\}$$

So, for example any Rmn. metric (Q, g) on Q

has its geodesic flow on $TQ \overset{g}{\longleftrightarrow} T^*Q$ given by a Hamiltonian system $H : T^*Q \to \mathbb{R}$, $H = \frac{\|p\|^2}{2}$, and the 'manifold of oriented geodesics' is a symplectic manifold (provided the quotient is a mfld.) So for example:

$\{\text{oriented geodesics on } S^2\} \approx S^2$



whereas for example $\{\text{oriented geodesics on } \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2\}$ are not a mfld.

Note that the space of oriented lines in $\mathbb{R}^3$, as we have already seen, are then a symplectic manifold as a special case of symplectic red.

As another example / extended exercise we have (regular) co-adjoint orbits $\mathcal{O}_\mu \subset \mathfrak{g}^*$ are symplectic manifolds, which one can see using symplectic reduction as follows:

① Let $G \circlearrowright G$ by <u>left</u> translations:

$$\varphi_g(h) = gh = L_g(h)$$

and consider its co-tangent lift:

$$G \circlearrowright T^*G, \quad \hat{L}_g(h, \alpha_h) = (gh, \alpha_h \circ dL_{g^{-1}}).$$

② Identify

$$T^*G \longleftrightarrow G \times \mathfrak{g}^*$$

$$(h, \alpha_h) \longleftrightarrow (h, \alpha_h \circ dR_h)$$

by <u>right</u> translations.

③ Let λ be the canonical 1-form on T^*G, and $\hat{X}_\xi = \frac{d}{dt}\Big|_0 \hat{L}_{\exp(\xi t)}$ the infinitesimal generators of the lifted action of left translation on T^*G.

check that, under the identification ②, the G-equivariant moment map, $\mu_\xi = \iota_{\hat{X}_\xi}\lambda$, is given by:

$$\mu : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$
$$(g, \mu_0) \longmapsto \mu_0 .$$

④ Conclude that $M_{\mu_0} = G \times \{\mu_0\} \simeq G$, and so

$\overline{M}_{\mu_0} = G_{\mu_0} \backslash G$ are the (left) cosets $G_{\mu_0} \cdot h$.

⑤ Identify $\overline{M}_{\mu_0} \simeq \mathcal{O}_{\mu_0} = \{ Ad^*_{g^{-1}} \mu_0 : g \in G \} \subset \mathfrak{g}^*$.

by

$$\begin{array}{ccc} G_{\mu_0} \cdot h & \longleftrightarrow & Ad^*_{h^{-1}} \mu_0 \\ \cap & & \cap \\ \overline{M}_{\mu_0} = G_{\mu_0} \backslash G & & \mathcal{O}_{\mu_0} \end{array}$$

so that the (regular) co-adjoint orbits $\mathcal{O}_{\mu_0} \subset \mathfrak{g}^*$ are symplectic manifolds.

⑥ The reduced symplectic structure $\overline{\omega}_{\mu_0} = \Omega$ on $\mathcal{O}_{\mu_0}$ can be given explicitly as follows. Let $\nu \in \mathcal{O}_{\mu_0}$ and $v_1 = \frac{d}{dt} Ad^*_{\exp(-\xi_1 t)} \nu$, $v_2 = \frac{d}{dt}\Big|_0 Ad^*_{\exp(-\xi_2 t)} \nu \in T_\nu \mathcal{O}_{\mu_0}$

then $\Omega_\nu (v_1, v_2) = \nu ([\xi_1, \xi_2])$.

Remark: There is on $\mathfrak{g}^*$ a canonical Poisson structure by for $f_1, f_2 : \mathfrak{g}^* \to \mathbb{R}$ taking $\{f_1, f_2\} : \mathfrak{g}^* \to \mathbb{R}$ through

$\{f_1, f_2\}(\mu) := \mu ([d_\mu f_1, d_\mu f_2])$, where we consider

$d_\mu f_i : \mathfrak{g}^* \to \mathbb{R}$ as elements of $\mathfrak{g}^{**} = \mathfrak{g}$.

The Poisson brackets of $\mathcal{O}_{\mu_0}$, $\omega_{\mu_0} = \Omega$ are related through $\{\bar{f}_1, \bar{f}_2\}_\Omega = \{f_1, f_2\}\big|_{\mathcal{O}_{\mu_0}}$ where $f_i\big|_{\mathcal{O}_{\mu_0}} = \bar{f}_i$ (one calls $\mathcal{O}_{\mu_0}$ a 'symplectic leaf' in the Poisson manifold $\mathfrak{g}^*, \{\cdot,\cdot\}$).

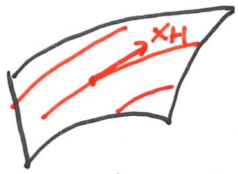
Finally, let us mention some dynamical situations in which symplectic reduction is relevant, namely: for Hamiltonian systems with symmetries.

$\boxed{\text{Prop}}$: For a Hamiltonian $H: M \to \mathbb{R}$, suppose $G \curvearrowright M$ is a Hamiltonian action by symmetries of H:

$$H(g\cdot x) = H(x) \qquad [\text{ie } H \text{ is } G\text{-invariant}].$$

Let $\mu: M \longrightarrow \mathfrak{g}^*$ be the moment map of $G \curvearrowright M$. Then

① $X_H \in TM_{\mu_0}$ is tangent to level sets of μ (ie μ are conserved quantites — or integrals — of X_H).



$M_{\mu_0} = \{\mu = \mu_0\}$

② Suppose $\bar{M}_{\mu_0} = M_{\mu_0}/G_{\mu_0}$ is a symplectic reduction (in particular the quotient is a manifold) with $\bar{\omega}_{\mu_0}$.

Then the trajectories of X_H ($\dot{\gamma} = X_H(\gamma)$) (lying in M_{μ_0}) project under $\pi: M_{\mu_0} \longrightarrow \bar{M}_{\mu_0}$ to trajectories of the 'reduced' Hamiltonian system

$\bar{H}_{\mu_0}: \bar{M}_{\mu_0} \longrightarrow \mathbb{R}$, $(\bar{M}_{\mu_0}, \bar{\omega}_{\mu_0})$ where

$$\begin{tikzcd} M_{\mu_0} \arrow[r, hook, "\iota"] \arrow[d, "\pi"'] & M \\ \bar{M}_{\mu_0} \end{tikzcd} \qquad \pi^* \bar{H}_{\mu_0} = \iota^* H = H\big|_{M_{\mu_0}}.$$

prf: for 1 it is a case of 'Noether theorem' (symmetries $\leftrightarrow$ integrals): 21

$$0 = \frac{d}{dt}\Big|_0 H(\exp(\xi t)\cdot x) = d_x H(X_\xi(x)) \overset{(*)}{=} \omega_x(X_\xi(x), X_H(x))$$

$$= -\omega_x(X_H(x), X_\xi(x)) = -d_x \mu_\xi(X_H(x)) = -\frac{d}{dt}\Big|_0 \mu_\xi(\varphi_t(x))$$

for φ_t the flow of X_H. So $\mu_\xi(\varphi_t(x)) = \text{cst.}$ $\forall \xi$ and in particular $\mu(\varphi_t(x)) = \text{cst.}$ [Alternately, from $(*)$ we have:

$$X_H(x) \in (T_x(G\cdot x))^\omega = T_x M_{\mu_0}\,].$$

for 2, let $v \in T_x M_{\mu_0}$ with $\pi_* v = \bar{v} \in T_{\bar{x}} \bar{M}_{\mu_0}$ $(\bar{x} = \pi(x))$.

Then, for $\bar{H}\circ\pi = H|_{M_{\mu_0}}$, we have:

$$d_x H(v) = \omega_x(v, X_H(x)) = \bar{\omega}_{\bar{x}}(\bar{v}, \pi_* X_H(x))$$

and:

$$d_x H(v) = d_{\bar{x}} \bar{H}(\bar{v}) = \bar{\omega}_{\bar{x}}(\bar{v}, X_{\bar{H}}(\bar{x})). \;[\text{since } H|_{M_{\mu_0}} = \bar{H}\circ\pi]$$

so that $\pi_* X_H = X_{\bar{H}}$. $\square$