Symplectic geometry 3: Some symplectic manifolds

* a 'leftover' exercise from last time:

recall in describing the projection, $\mathrm{Sym}^2_+(V) \to \mathcal{J}(V,\omega)$, we found that any inner product (pos. def.) g on V can be written as:

$$\omega(u,v) = g(u, PJv)$$

for $J \in \mathcal{J}(V,\omega)$ and P symmetric, pos. def with $JP = PJ$.

(1) Show there exists a symplectic basis $V \simeq \mathbb{R}^{2n} \ni (q,p)$ in which the solid ellipsoid

$$B_g = \{ g(v,v) \leq 1 \} \subset V$$

is given by:

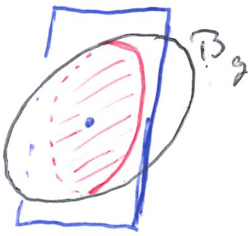
$$\left\{ \frac{p_1^2 + q_1^2}{r_1^2} + \cdots + \frac{p_n^2 + q_n^2}{r_n^2} \leq 1 \right\}$$

for some $0 < r_1 \leq \cdots \leq r_n$ (the 'symplectic spectrum' of B_g).

(2) Show that the critical values of the map

$$S : Gr(2,V) \longrightarrow \mathbb{R}$$
$$\Pi \longmapsto \mathrm{Area}_\omega(\Pi \cap B_g)$$

are $\pm \sigma_j = \pi r_j^2$.



$\Pi \subset V$
a 2-plane thru origin

Note: this is the 'symplectic' analogue of that in an inner product space $(V, \langle\cdot,\cdot\rangle)$, given any $g \in \mathrm{Sym}^2_+(V)$ there exists an orthonormal basis $V \simeq \mathbb{R}^n \ni x$ in which:

$$B_g = \{ g(v,v) \leq 1 \} \longleftrightarrow \left\{ \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1 \right\}$$

and the critical values of the map:

$\{g(v,v)=1\} = \mathcal{E}_g \longrightarrow \mathbb{R}_+$, $v \longmapsto |v|$ are the a_j's.

Basic questions on symplectic manifolds:

1. Given an even dimensional mfld M, does there exist a symplectic structure on M?

2. Given two symplectic structures ω_0, ω_1 on M, when are they equivalent? ($\varphi^*\omega_1 = \omega_0$ some diffeo $\varphi: M \circlearrowleft$).

Def. We call $(M,\omega), (M',\omega')$ symplectic manifolds <u>equivalent</u> (or 'symplectomorphic') if there is a diffeo $\varphi: M \to M'$ s.t. $\varphi^*\omega' = \omega$, and write $(M,\omega) \sim (M',\omega')$.

First let's consider symplectic structures on a closed, oriented surface Σ.

Any such surface has symplectic structures (take any areaform).

Note that if ω_0, ω_1 are two (commonly oriented) area forms on Σ, then if we have $\varphi: \Sigma \circlearrowleft$ with $\varphi^*\omega_1 = \omega_0$ then:

$$\int_\Sigma \omega_0 = \int_\Sigma \omega_1 .$$

This condition is also sufficient:

Theorem (Moser) Let ω_0, ω_1 two commonly oriented area forms on Σ. Then there exists a diffeo $\varphi: \Sigma \circlearrowleft$ with $\varphi^*\omega_1 = \omega_0$ iff

$$\int_\Sigma \omega_1 = \int_\Sigma \omega_0 .$$

<u>prf</u>: ($\Leftarrow$) Supposing the two symplectic forms on Σ have the same total area, we have $[\omega_1 - \omega_0] = 0 \in H^2(\Sigma)$, so that $\omega_1 - \omega_0 = d\alpha$ for some 1-form $\alpha \in \Omega^1(M)$. Set

$$\omega_t := (1-t)\omega_0 + t\omega_1 = \omega_0 + t\, d\alpha \quad \text{for } t \in [0,1].$$

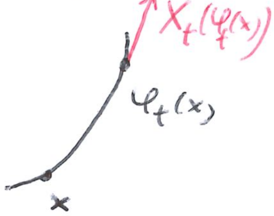
Since ω_0, ω_1 are commonly oriented, ω_t is also an area form (a symplectic structure on Σ) for each $t \in [0,1]$.

we will construct a 1-param. family of diffeos

$$\varphi_t : \Sigma \circlearrowleft \qquad t \in [0,1] \quad , \quad \varphi_0 = id$$

such that $\varphi_t^* \omega_t = \omega_0$ (in particular $\varphi_1^* \omega_1 = \omega_0$).

Let X_t be the (time dependent) vectorfields generating φ_t:



$$X_t(\varphi_t(x)) = \frac{d}{dt} \varphi_t(x) \left(= \frac{d}{d\varepsilon}\Big|_0 \varphi_{t+\varepsilon}(x) \right).$$

By Cartan's formula (w/ time dependence):

$$\frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* \left(\iota_{X_t} \overset{0}{\cancel{d\omega_t}} + d(\iota_{X_t}\omega_t) + \frac{d}{dt}\omega_t \right)$$

$$= \varphi_t^* \left(d(\iota_{X_t}\omega_t) + d\alpha \right) = d\left[\varphi_t^* (\iota_{X_t}\omega_t + \alpha) \right].$$

So if we choose X_t through: $\iota_{X_t}\omega_t = -\alpha$,

then we may integrate $\dot{x} = X_t(x)$ to have $\varphi_t : \Sigma \circlearrowleft$

for $t \in [0,1]$ (since Σ is compact the 'flow' of X_t is defined for all $t \in [0,1]$) with $\varphi_0 = id$ and $\frac{d}{dt}(\varphi_t^* \omega_t) = 0$, in particular

$$\varphi_0^* \omega_0 = \omega_0 = \varphi_t^* \omega_t = \varphi_1^* \omega_1 . \qquad \square$$

We can use this same technique 'Moser's trick' in other situations:

$\boxed{\text{Theorem}}$ (DARBOUX normal form): Let $x \in (M, \omega)$, then there exists a neighborhood $x \in U \subset M$ and chart

$$\varphi : U \to \mathbb{R}^{2n} \qquad \text{with} \qquad \varphi^*(dp \wedge dq) = \omega .$$

pf: Consider some local coordinates centered at x:

$$\psi : \mathbb{R}^{2n} \to U \subset M \qquad \psi(0) = x$$

and take a linear change of coords so that

$$\psi^* \omega_x = dp \wedge dq .$$

Set $\omega_1 = \psi^* \omega$ and $\omega_0 = dp \wedge dq$, so that

$$\omega_1|_0 = \omega_0|_0 \,(= \omega_0).$$

Since we are on $\mathbb{R}^{2n}$, $\omega_1 - \omega_0 = d\alpha$ is exact, and WLOG $\alpha|_0 = 0$ (by adding suitable df to α if necessary).

Take $\omega_t = \omega_0 + t\, d\alpha$

which (since $\omega_1 - \omega_0|_0 = 0 = d\alpha|_0$) is non-degenerate still for $t \in [0,1]$ and over some neighbrohood $0 \in \mathcal{U}_0 \subset \mathbb{R}^{2n}$.

The vector field X_t through $\iota_{X_t}\omega_t = -\alpha$ then has $\frac{d}{dt}\varphi_t^*\omega_t = 0$ as long as its flow φ_t is defined.

Since $X_t(0) = 0$, this flow is defined for $t \in [0,1]$ and initial condition in some sufficiently small nbhd $0 \in \mathcal{U}_1 \subset \mathcal{U}_0 \subset \mathbb{R}^{2n}$ of the origin, and so $\varphi_1^*\omega_1 = \omega_0$ and for the chart

$$\begin{tikzcd} M \supset \mathcal{U} \arrow[r, "\psi"] \arrow[rr, bend right, "\varphi"'] & \mathbb{R}^{2n} \supset \mathcal{U}_1' \arrow[r, "\varphi_1^{-1}"] & \mathcal{U}_1 \end{tikzcd}$$

we have $\varphi^*\omega_0 = \omega$. $\square$

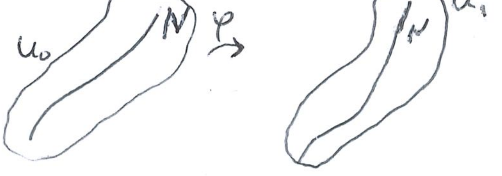
And more generally:

$\boxed{\text{Theorem}}$ (Darboux/Weinstein Neighorhood normal forms)

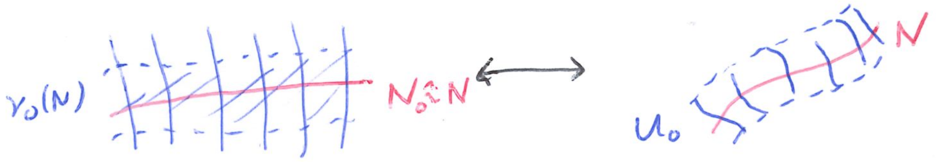
Let $N \subset (M, \omega_0)$ a compact submanifold and suppose ω_1 is another symplectic structure on M such that

$$(\omega_0)_x(u,v) = (\omega_1)_x(u,v) \quad \forall x \in N,\ u,v \in T_x M$$

($\omega_0|_{T_N M} = \omega_1|_{T_N M}$). Then there exist neighborhoods $\mathcal{U}_1 \supset N$, $\mathcal{U}_0 \supset N$ of N in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*\omega_1|_{\mathcal{U}_1} = \omega_0|_{\mathcal{U}_0}$ and $\varphi|_N = id_N$ ($\varphi(x) = x \ \forall x \in N$).



prf: Let $\nu(N) = T_N M / TN$ be the normal bundle of $N \subset M$. Choosing some arbitrary Rmn. metric on M, we may identify a neighborhood $\nu_0(N) \subset \nu(N)$ of the zero section with a neighborhood $\mathcal{U}_0 \supset N$ of N in M:



(use the metric to identify $\nu(N) = N^\perp \to N$ & exponential map (cpt'ness of N)).

In particular we have a family of retracting maps:

$$r_t : \mathcal{U}_0 \to \mathcal{U}_0 \;;\; r_t(n) = n \;\; \forall n \in N \;\&\; r_1 = id_{\mathcal{U}_0} \;\&$$

$$r_0 : \mathcal{U}_0 \to N$$

being, under the correspondence above $\underset{\nu_0(N)}{\underset{\Downarrow}{(n, \nu_n)}} \to \underset{\nu_0(N)}{\underset{\Downarrow}{(n, t\nu_n)}}$.

By Cartan's formula we have for any k-form μ on $\mathcal{U}_0$:

$$(*) \quad \mu - r_0^* \mu = \int_0^1 \frac{d}{dt}(r_t^* \mu)\, dt = \int_0^1 r_t^* \left(\iota_{\xi_t} d\mu + d(\iota_{\xi_t} \mu) \right) dt$$

for $\xi_t(r_t(x)) = \frac{d}{dt} r_t(x)$. In particular, for

$$\omega_t := \omega_0 + t\beta \;,\; \beta = \omega_1 - \omega_0$$

we have, since $\beta|_N = 0$ & $d\beta = 0$ that $(*)$ reads

$$\beta = d\left(\int_0^1 r_t^*(\iota_{\xi_t} \beta)\, dt \right) = d\alpha$$

where $\alpha = \int_0^1 r_t^*(\iota_{\xi_t}\beta)\, dt$ has $\alpha|_N \equiv 0$. Hence again:

$$\omega_t = \omega_0 + t\, d\alpha$$

and since $d\alpha|_N = \beta|_N = 0$ there is some (perhaps smaller) nbhd $\mathcal{U}_0 \supset V \supset N$ on which ω_t is non-degenerate for $t \in [0,1]$, so we may choose X_t through $\iota_{X_t}\omega_t = -\alpha$

where, since $\alpha|_N = 0$ we have $X_t|_N \equiv 0$, so that the flow φ_t of X_t has $\varphi_t(n) = n$ $\forall n \in N$ and $\varphi_t^* \omega_t = \omega_0$. Again since $X_t|_N \equiv 0$ and N is compact, there is some sufficiently small nbhd $U' \supset N$ on which φ_t is defined for all $t \in [0,1]$ so that $\varphi_1 : U' \to U''$ has $\varphi_1^* \omega_1 = \omega_0$. $\square$.

Remark: Darboux's normal form is a special case of this neighborhood thm when we take $N = \{pt.\}$.

Example: Let us return to the 1st question on when M^{2n} may admit a symplectic structure. As an example, we can note that any even dimensional sphere S^{2n}, $n > 1$ does not admit any symplectic structure: if it did we would have some $[\omega] \in H^2(S^{2n}) = 0$ with $0 \neq [\omega^n] \in H^{2n}(S^{2n}) \cong \mathbb{R}$ (non-degen) $\Rightarrow \int_{S^{2n}} \omega^n \neq 0$

but if $[\omega] = 0$, ie $\omega = d\lambda$ is ~~closed~~ exact then so is $[\omega^n] = 0$ ~~closed~~ exact, and $\int_{S^{2n}} \omega^n = 0$.

It is a deeper theorem that a sphere S^{2n} admits an almost-complex structure J iff $n = 1$ or 3 (Borel-Serre). on S^2 we have a complex structure (as the Riemann sphere), and it is an open question whether there exists a complex structure on S^6.

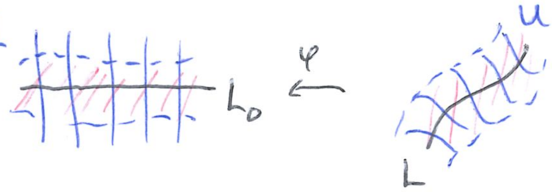
Lagrangian neighborhoods

As a special case of the Weinstein neighborhood theorem:

Theorem (Weinstein) Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold. Then there is a neighborhood $L \subset U \subset M$ and diffeomorphism $\varphi : U \to U_0 \subset T^*L$

for $U_0 \subset T^*L$ a neighborhood of the zero section such that

$$\varphi^* d\lambda_L = \omega|_{U_0}$$



for λ_L on T^*L the canonical 1-form.

prf: Note that we can always identify (for L Lagrangian):

$$\nu(L) = T_L M / TL = T^*L \qquad \text{through}$$

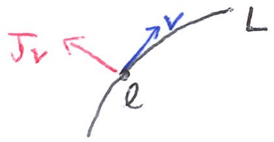
(*) $$\nu + T_\ell L \longleftrightarrow \left(T_\ell L \ni v \mapsto \omega_\ell(\nu, v) \right).$$

Now choose some $J \in \mathcal{J}(M,\omega)$ with associated Riem. metric

$$g_J(u,v) := \omega(Ju, v).$$

Then we have an identification: (orthogonal complement wrt g_J).

$$T^*L = \nu(L) \approx J(TL) = TL^\perp$$



through:

$$\underset{(v \in T_\ell L)}{Jv} \xleftrightarrow{(**)} \left(T_\ell L \ni u \overset{v^*}{\longmapsto} \omega_\ell(Jv, u) \right)$$

or for short

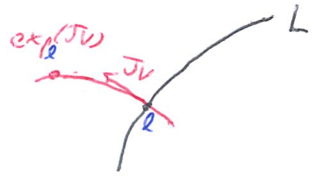
$$\begin{array}{ccc} Jv & \xleftrightarrow{(**)} & v^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ J(TL) = TL^\perp & & T^*L \end{array}.$$

[NOTE: $J(T_\ell L) \subset T_\ell M$ is Lagrangian subspace & $T_\ell M = T_\ell L \oplus J(T_\ell L)$ w/ $J(T_\ell L) = (T_\ell L)^\perp$ the $\perp$ wrt g_J.]

So we consider the g_J-exponential map:

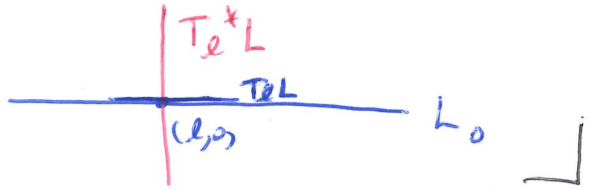
$$T^*L \xrightarrow{\varphi} M$$

$$(\ell, v^*) \longmapsto \exp_\ell(Jv)$$



(where $v^* \longleftrightarrow Jv$ through (**)).

NOTE: $T_{(\ell,0)}(T^*L) = T_\ell L_0 \oplus T_\ell^* L = T_\ell L \oplus T_\ell^* L$:



and we compute:

1) $d\varphi_{(\ell,0)}(u,0) = u$, for $u \in T_\ell L$

2) $d\varphi_{(\ell,0)}(0,v^*) = Jv$, for $v^* \in T_\ell^* L$,

so that: $(\varphi^*\omega)_{(\ell,0)}((u,v^*),(u_1,v_1^*)) = \omega_\ell(u+Jv, u_1+Jv_1)$

$$= \omega_\ell(Jv,u_1) - \omega_\ell(Jv_1,u) = v^*(u_1) - v_1^*(u)$$

$$= (\omega_L)_{(\ell,0)}((u,v^*),(u_1,v_1^*))$$

for $\omega_L = d\lambda_L$ the standard symplectic structure on T^*L.

Hence $(\varphi^*\omega)|_{L_0} = \omega_L|_{L_0}$, and so by the nbhd thm, there is some neighborhood of the zero section of T^*L on which $\varphi^*\omega = \omega_L$. □.

Lagrangians and fixed points

Let (M,ω) be a symplectic manifold and consider the symplectic manifold

$$M\times M,\ \Omega = \pi_1^*\omega - \pi_2^*\omega$$

$(\pi_j(m_1,m_2) = m_j)$

The graph of a diffeo $f: M \circlearrowleft$

$$\Gamma_f = \{(m,f(m)) : m\in M\} \subset M\times M$$

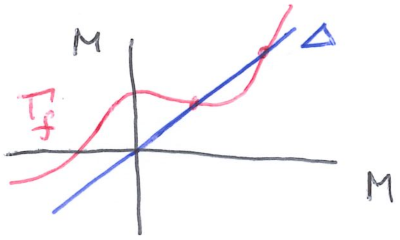
is a Lagrangian submanifold of $M\times M, \Omega$ iff $f^*\omega=\omega$ is a symplectomorphism of (M,ω).

In particular, the diagonal (graph of identity function)

$$\Delta = \{ (m,m) : m \in M \} \subset M \times M$$

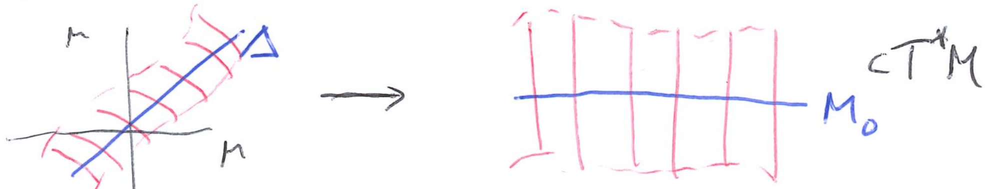
is a Lagrangian submanifold of $M \times M, \Omega$.

Fixed points of a symplectomorphism $f : M \circlearrowleft$ $f^*\omega = \omega$ are then equivalent to intersection points of the Lagrangian submanifolds $\Gamma_f, \Delta \subset M \times M, \Omega$

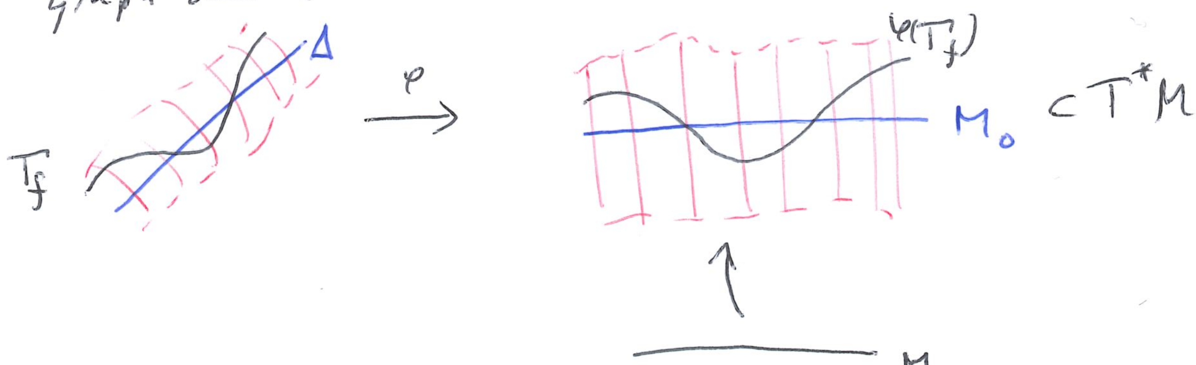


So that general results about intersection #s of Lagrangian submanifolds are interesting: $\#(L_0 \cap L_1)$ is a generalization of counting fixed pts. of symplectic maps.

In certain cases these normal form theorems can give us some partial information. Consider that $\Delta \cong M$ so that we have a neighborhood of the Lagr. submfld $\Delta \subset M \times M$ symplectomorphic to $T^*\Delta \cong T^*M$



a symplectomorphism $f : M \circlearrowleft$ sufficiently close to the identity has its graph Γ_f corresponding to a Lagrangian submanifold of T^*M which is a graph over the zero section (M_0), ie a closed 1-form on M:



$\varphi(\Gamma_f) = \mathrm{im}(\beta)$ for $\beta : M \to T^*M$ a closed 1-form.

In particular if $H^1(M)=0$ (every closed 1-form on M is exact) then any such $f: M \circlearrowleft$, $f^*\omega=\omega$, corresponds to a graph $\mathrm{im}(dS) \subset T^*M$ for some $S: M \to \mathbb{R}$.

The intersection with the zero section (the fixed points of f) correspond to critical points of S.

For example, we find:

$\boxed{\text{Example:}}$ Let $f: S^2 \circlearrowleft$ an orientation and area preserving map $f^*\omega=\omega$ some area form ω on S^2. Then f has at least 2 fixed points (provided it is sufficiently close to the identity).

Examining to what extent the 'sufficiently close to the identity' hypothesis may be dropped is a main theme in symplectic geometry (Arnold conjectures).