Symplectic Geometry 2: Linear symplectic geometry

Def. A symplectic vector space is a ($2n$-dim.) vector space V with a non-degenerate, skew, bi-linear form $\omega : V \times V \to \mathbb{R}$.

Example: $\mathbb{R}^{2n} \ni (q,p)$, $\omega = dp \wedge dq$. Note that the basis $\partial_{q_1}, \dots, \partial_{q_n}, \partial_{p_1}, \dots, \partial_{p_n}$ has:

$$\omega(\partial_{q_j}, \partial_{q_k}) = 0 = \omega(\partial_{p_j}, \partial_{p_k}), \quad \omega(\partial_{p_j}, \partial_{q_k}) = \delta_{jk}$$

called a symplectic basis (the symplectic analogue of orthonormal basis for an inner product).

In this basis, we have the matrix representation:

$$\omega(u,v) = u \cdot Jv, \quad J = \begin{pmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix} \quad (u, v \in \mathbb{R}^{2n})$$

(where $J\partial_{q_j} = \partial_{p_j}$, $J\partial_{p_j} = -\partial_{q_j}$, w/ $J^2 = -Id$ is a complex structure on $\mathbb{R}^{2n}$). Identifying

$$\begin{aligned} \mathbb{R}^{2n} &\longleftrightarrow \mathbb{C}^n \\ (q,p) &\longleftrightarrow Z = (z_1, \dots, z_n) \qquad z_j = q_j + i p_j \end{aligned}$$

then $J(q,p) \longleftrightarrow iZ$, and $\omega(u,v) = u \cdot Jv \longleftrightarrow \omega(Z,W) = Z \cdot (iW)$.

Note that for the Hermitian product:

$$\langle Z, W \rangle = Z_1 \overline{W_1} + \dots + Z_n \overline{W_n} \quad \text{on } \mathbb{C}^n,$$

its real/imaginary parts are:

$$\langle Z, W \rangle = Z \cdot W + i\, \omega(Z, W).$$

Example: Let $\mathcal{U}$ be some n-dim. vector space and $V = \mathcal{U} \times \mathcal{U}^*$, with symplectic structure

$$\omega((u,u^*),(v,v^*)) = u^*(v) - v^*(u).$$

(note $\mathcal{U} \times \mathcal{U}^* = T^*\mathcal{U}$)

Then for any basis $e_1, \dots, e_n$ of U, let $f_1 = e^1, \dots, f_n = e^n$ be the corresponding dual basis of U^*.
The basis $e_1, \dots, e_n, f_1, \dots, f_n$ of $V = U \times U^*$ is a symplectic basis:
$\omega(e_j, e_k) = 0 = \omega(f_j, f_k)$, $\omega(f_j, e_k) = \delta_{jk}$.

Example: For (V, ω) a sympl. vect. space consider on $V \times V$
$\Omega = \pi_1^*\omega - \pi_2^*\omega$ $[\pi_j(v_1, v_2) = v_j]$. The use of the '$-$' sign is more interesting b/c a graph $\Gamma_f = \{(v, f(v)) : v \in V\} \subset V \times V$ of $f: V \to V$ is a Lagrangian submanifold of $(V \times V, \Omega)$ exactly when $f^*\omega = \omega$ is a symplectomorphism of V.

Remark: Darboux's theorem states that any symplectic manifold (M, ω) is locally equivalent to a symplectic vector space (we have seen this with cotangent bundles). Linear symplectic geometry plays a more fundamental role in symplectic geometry than inner product spaces in Riemannian geometry.

Subspaces, Complements

Def: Let $U \subset (V, \omega)$ a subspace. Its symplectic complement is:

$$U^\omega = \{v \in V : \omega(v, u) = 0 \quad \forall u \in U\}.$$

Example: The symplectic complement of U may intersect U nontrivially. For example if $U \subset V$ is a line ($\dim U = 1$), then $U \subset U^\omega$.

Prop: $\dim U + \dim U^\omega = \dim V$ $(= 2n)$.

pf: Let $Ann(U) = \{\alpha \in V^* : U \subset \ker \alpha\}$. Then

- $\dim Ann(U) = \dim V - \dim U$
- $b^{-1}(Ann(U)) = U^\omega$, where $b: V \to V^*$, $v \mapsto \iota_v \omega$ is invertible (ω is non-degenerate). $\square$

$\boxed{\text{Def.}}$ Call $U \subset V$ a

1) symplectic subspace if $U \cap U^\omega = 0$,
2) isotropic subspace if $U \subset U^\omega$,
3) co-isotropic subspace if $U^\omega \subset U$,
4) Lagrangian subspace if $U = U^\omega$.

* note: a Lagrangian subspace then has dimension $n = \frac{1}{2}\dim V$ & $\omega|_U \equiv 0$ *

$\boxed{\text{Examples:}}$ in $\mathbb{R}^{2n} \ni (q,p) \longleftrightarrow Z \in \mathbb{C}^n$ w/ $\omega = dp \wedge dq$,

1) $(q_1, \dots, q_k, 0, \dots, 0, p_1, \dots, p_k, 0, \dots, 0) \longleftrightarrow \mathbb{C}^k \times \{0\}$ is a symplectic subspace.

2) $(q_1, \dots, q_k, 0, \dots, 0) \longleftrightarrow \mathbb{R}^k \times \{0\}$ is an isotropic subspace (Lagrangian if $k = n$).

3) $(q_1, \dots, q_k, 0, \dots, 0, p_1, \dots, p_n) \longleftrightarrow \mathbb{C}^k \times \mathbb{R}^{n-k}$ is a co-isotropic subspace (Lagrangian if $k = 0$).

$\boxed{\text{Exercise:}}$ give an example of a subspace $U \subset (V, \omega)$ that is <u>not</u> symplectic, isotropic, co-isotropic (or Lagrangian). /

$\boxed{\text{Prop.}}$ 1) $U \subset W \Rightarrow W^\omega \subset U^\omega$,

2) $(U^\omega)^\omega = U$,

3) $(U \cap W)^\omega = U^\omega + W^\omega$ (equivalent by (2) to $U^\omega \cap W^\omega = (U+V)^\omega$)

<u>prf:</u> we will prove (3) (leaving (1), (2) as verifications): let $A, B \subset V$ some subspaces. We have always:

(*) $\quad (A+B)^\omega \subset A^\omega \cap B^\omega$

indeed, if $\omega(v, a+b) = 0 \ \forall a, b \in A, B$ then in particular

$\omega(v, a) = 0 \ \forall a \in A$ (take $b=0$) and $\omega(v, b) = 0 \ \forall b \in B$ (take $a=0$),

so that $v \in (A+B)^\omega \Rightarrow v \in A^\omega \cap B^\omega$.

On the other hand, we also have, for any $A, B \subset V$ that

$A \cap B \subset A, B \overset{(1)}{\Rightarrow} A^\omega, B^\omega \subset (A \cap B)^\omega \Rightarrow A^\omega + B^\omega \subset (A \cap B)^\omega$. (**) ④

Now, consider $U, W < V$. By (**) with $A = U, B = W$, we have:

$$(U \cap W)^\omega \supset U^\omega + W^\omega \quad (*)$$

By (*) with $A = U^\omega$, $B = V^\omega$, and (2) we have:

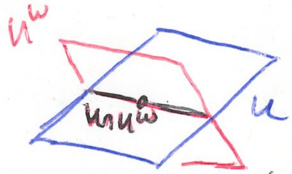
$$(U^\omega + W^\omega)^\omega \subset U \cap W \overset{(1)}{\Rightarrow} U^\omega + W^\omega \supset (U \cap W)^\omega \quad (**)$$

so $(*), (**) \Rightarrow (U \cap W)^\omega = U^\omega + W^\omega$ as claimed. $\square$.

$\boxed{\text{Prop}}$: ('linear symplectic reduction')

1) $U / U \cap (U^\omega)$ is a symplectic vector space (in particular even dimensional).

2) If U is co-isotropic: $U \supset U^\omega$, then for any Lagrangian $L \subset V$, $\overline{L} = (L \cap U)/U^\omega \subset U/U^\omega$ is also Lagrangian.



(do not take dimensions literally)

prf: i) define symplectic form $\overline{\omega}$ on $U / U \cap (U^\omega)$ by $\overline{\omega}(u_1 + W, u_2 + W) := \omega(u_1, u_2)$ $(W = U \cap U^\omega \subset U)$

check: $\overline{\omega}$ is well defined & non-degen on U/W.

2) Set $\overline{U} = U / U^\omega$ in this co-isotropic case, with $\overline{\omega}$ from (1).

Sps. $\overline{\omega}(\overline{x}, \overline{l}) = 0 \quad \forall \overline{l} \in \overline{L}$ $(\overline{x} \in \overline{L}^{\overline{\omega}})$. Write $\overline{x} = x + U^\omega$ some $x \in U$, then by def of $\overline{\omega}$, we have:

$$\omega(x, v) = 0 \quad \forall v \in L \cap U, \text{ ie } x \in (L \cap U)^\omega = L + U^\omega.$$

or: $x = l + u'$ some $l \in L$ and $u' \in U^\omega \subset U$.

but $x \in U$, so that $l = x - u' \in L \cap U$, hence:

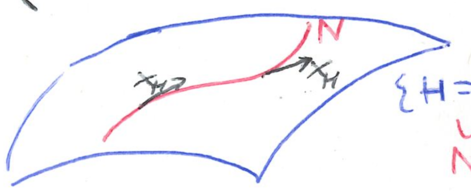
$\overline{x} = l + U^\omega \in \overline{L} = L \cap U / U^\omega$ $(l \in L \cap U)$, so that $\overline{L}^{\overline{\omega}} \subset \overline{L}$. ~~Also~~ for the other direction, note that $\overline{L}$ is always isotropic: $\overline{\omega}|_{\overline{L}} \equiv 0$ so that $\overline{L} \subset \overline{L}^{\overline{\omega}}$. $\square$

$\boxed{\text{Example}}$: Any hyperplane $U \subset V$ ($\dim U = \dim V - 1 = 2n-1$), has symplectic complement a line: $l_U = U^\omega \subset U$ ($\dim l_U = 1$).

If $\Sigma \subset (M, \omega)$ is a $2n-1$ dim submanifold ($\dim M = 2n$), then $(x \in \Sigma)$ $(T_x\Sigma)^{\omega_x} = l_x \subset T_x\Sigma$ is a line field on Σ.

* when $\Sigma = \{H = c\}$ is a regular level set of some $H : M \to \mathbb{R}$, then this line field is directed by the symplectic gradient X_H. *

Similarly, suppose $N \subset \Sigma$ is a Lagrangian submanifold contained in a regular hypersurface Σ. Then $l_x \subset T_x N$ $\forall x \in N \subset \Sigma$, i.e. this Lagr. submanifold $N \subset \Sigma$ is invariant under the flow of any vector field spanning l_x: use the linear algebra result that if $U \subset V$ is a hyperplane with $l_U = U^\omega$ and $W \subset U$ is Lagrangian then $W \oplus l_U$ is Lagrangian (and has dimension at most $n = \dim W$).



$\{H = cst\} \subset M$ $(H : M \to \mathbb{R})$ $\Rightarrow$ $X_H \in TN$.
$\cup$
N (Lagrangian)

("Linear DARBOUX"): Let (V, ω) be a symplectic vector space. Then V has a symplectic basis (in particular, we have a linear ~~symplecto~~ symplectomorphism $(V, \omega) \to (\mathbb{R}^{2n}, dp \wedge dq)$).

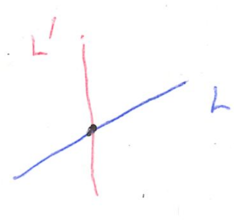
$\boxed{\text{prf:}}$ * remark: there is an inductive proof as well, we will give a different proof that has slightly more information. *

$\boxed{\text{Claim 1}}$: any symplectic vector space has a Lagrangian subspace:

$L \subset V$

$\boxed{\text{Claim 2}}$: any Lagrangian subspace $L \subset (V, \omega)$ has a Lagrangian complement $L' \subset V$ Lagrangian with $L \cap L' = 0$ ($\Rightarrow V = L \oplus L'$)

(* note L' is not unique *)



Claim 3: for $L' \subset V$ any Lagrangian subspace, then

$$(V/L')^* \simeq L'.$$

from these claims the proposition follows, since we write (1,2): $V = L \oplus L'$, and by 3 have $L' \simeq (V/L')^* = L^*$, so that $V \simeq L \times L^*$. Then we check that the symplectic form ω on V under these identifications is the standard one:

(*) $\omega((l, l^*), (l_1, l_1^*)) = l^*(l_1) - l_1^*(l) \quad (l \in L, l^* \in L^*)$.

Any basis for L w/ corr dual basis for L^* is then a symplectic basis for V identifying $(V, \omega) \simeq (\mathbb{R}^{2n}, dp \wedge dq)$.

Lets show then the above claims.

Claim 1: There always exists an isotropic subspace in V (e.g. any line).
Let $U \subset U^\omega$ be an isotropic subspace of V and take

$u' \in U^\omega \setminus U$ (assuming U is not Lagrangian)

$U' = U \oplus \langle u' \rangle$ then $u' \in (U')^\omega$ $\left(\begin{matrix} \omega(u', u) = 0 \ \forall u \in U \\ \Downarrow \\ \omega(u', u') = 0 \end{matrix} \right)$

and since $U' \subset U^\omega$, we have $U = (U^\omega)^\omega \subset (U')^\omega$ as well

so that $U' = U \oplus \langle u' \rangle \subset (U')^\omega$ and U' is also co-isotropic

(with $\dim U' = \dim U + 1$ if U is not already Lagrangian).

Continue adjoining such elements until $\dim U' = \frac{1}{2} \dim V$ is Lagrangian.

Claim 2: There always exists an isotropic subspace transverse to a given Lagr. subspace L (take ~~any~~ line $\ell \cap L = \{0\}$).
Let $U \subset U^\omega$ be a transverse isotropic subspace to L ($U \cap L = \{0\}$).
we would like to choose then a $u' \in U^\omega \setminus (L \cup U)$, so that $U' = U \oplus \langle u' \rangle \supset U$ is isotropic and $U' \cap L = \{0\}$. To see such a u' exists, we have (by 'linear sympl. reduction') that U^ω/U is symplectic and $(L \cap U^\omega)/U \subset U^\omega/U$ is a Lagrangian subspace (in particular we have u''s in $U^\omega \setminus (L \cup U)$ to choose).

finally for claim 3, we take

$$(*) \quad L' \ni l' \longleftrightarrow \left(v + L' \longmapsto \omega(l', v) \right) \in \left(V/L' \right)^*$$

Then for $V = L \oplus L'$ from (1), (2) we have:

$$\omega(l + l', l_1 + l_1') = \omega(l', l_1) - \omega(l_1', l) \quad (l, l_1 \in L, l', l_1' \in L')$$

which under identification $(*)$ $\quad l' \longleftrightarrow \big(l^*(\underset{l \in L}{l}) = \omega(l', l)\big) \in L^*$

for $V \approx L \times L^*$ reads as stated in $(*)$. $\square$

Linear symplectic group

Def. a linear transformation $A : (V, \omega) \to (V, \omega)$ is called a linear symplectic transformation if

$$\omega(Au, Av) = \omega(u, v) \quad \forall u, v \in V \quad (A^*\omega = \omega).$$

The linear symplectic group of a sympl. v.s. V, ω is the group of all linear symplectic transformations of (V, ω), we denote it:

$$Sp_{2n}(V, \omega). \qquad (2n = \dim V)$$

Remark: by the linear Darboux theorem $(V, \omega) \approx (\mathbb{R}^{2n}, dp \wedge dq)$ and $Sp_{2n}(V, \omega) \approx Sp_{2n}(\mathbb{R}^{2n}, dp \wedge dq) =: Sp_{2n}(\mathbb{R})$.

First, lets count the dimension of this group. There are bijections:

$$Sp_{2n}(V, \omega) \longleftrightarrow \{ \text{symplectic bases of } V, \omega \}$$

(fix some symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ of V and associate each $A \in Sp_{2n}(V, \omega)$ with the symplectic basis $Ae_1, \dots, Ae_n, Af_1, \dots, Af_n$).

By the proof of the linear Darboux theorem, a symplectic basis of V amounts to the choice of a pair $L, L' \in \Lambda(n)$ of Lagrangian subspaces in V, as well as a choice of basis of L ($\dim L = n$). So we have:

$$(*) \qquad \dim Sp_{2n}(\mathbb{R}) = 2\dim\Lambda(n) + n^2$$

for $\Lambda(n)$ the Lagrangian grassmanian (eg of $\mathbb{R}^{2n}$, $dp \wedge dq$).

$\boxed{\text{Prop.}}$ $\dim\Lambda(n) = \frac{n(n+1)}{2}$ (so $\dim Sp_{2n}(\mathbb{R}) = n(2n+1)$).

prf: Identify $V \simeq L_0 \times L_0^*$, $\omega((l,l^*),(l_1,l_1^*)) = l^*(l_1) - l_1^*(l)$.
The (open set) of Lagrangian sub~~sets~~ spaces given by graphs:

$$L = \{(l, Al) : l \in L_0\}, \quad A : L_0 \to L_0^*$$

are parametrized by $A : L_0 \to L_0^*$ s.t.

$$0 = \omega((l, Al),(l_1, Al_1)) = (Al, l_1) - (Al_1, l)$$

$\Leftrightarrow$ $A^* = A$ is symetric. The $n \times n$ symetric matrices depend on $\frac{n(n+1)}{2}$ parameters. $\square$

We can also compute $\dim Sp_{2n}(\mathbb{R})$ more directly:

$\boxed{\text{Prop.}}$ $A : (\mathbb{R}^{2n}, dp \wedge dq) \circlearrowleft$ is in $Sp_{2n}(\mathbb{R})$ iff

$$A^t J A = J \qquad \left(J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right)$$

(explicitely, if we write $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ in 'block form', we have the conditions: $X^t Z = Z^t X$, $Y^t W = W^t Y$, $W^t X - Y^t Z = I_{n \times n}$).

prf: $u \cdot Jv = Au \cdot JAv = u \cdot A^t J A v \quad \forall u, v \in \mathbb{R}^{2n}$. $\square$

And for its Lie algebra:

$\boxed{\text{Prop.}}$ The Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$ of (the matrix group) $Sp_{2n}(\mathbb{R})$ is represented by $\xi : \mathbb{R}^{2n} \circlearrowleft$ s.t.

$$\xi^T J + J \xi = 0.$$

(explicitely, in 'block form' $\xi = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, the conditions: $z^t = z$, $y^t = y$, $w^t + x = 0$)

and may be identified with the $2n\times 2n$ symmetric matrices through $\mathfrak{sp}_{2n}(\mathbb{R}) \ni \xi \longleftrightarrow J\xi = B \in Sym_{2n}(\mathbb{R})$.

in particular we find, again, $\dim Sp_{2n}(\mathbb{R}) = \dim \mathfrak{sp}_{2n}(\mathbb{R}) = n(2n+1)$.

prf: we just differentiate $A(s)^t J A(s) = J$ at $s=0$ for $s \mapsto A(s) \in Sp_{2n}(\mathbb{R})$ w/ $A(0) = Id_{2n\times 2n}$ (considering $\mathfrak{sp}_{2n}(\mathbb{R}) \approx T_{Id} Sp_{2n}(\mathbb{R})$), and take $\xi = A'(0)$. Note that $J^t = -J$ so $J\xi$ is symmetric iff $\xi \in \mathfrak{sp}_{2n}(\mathbb{R})$. $\square$

Remark: equivalently the Lie algebra is identified with linear maps $\xi : V \to V$ s.t. $\omega(\xi u, v) + \omega(u, \xi v) = 0 \quad \forall u, v \in V$.
The corresponding symmetric matrices being represented by the quadratic forms $B_\xi(v) = \frac{\omega(v, \xi v)}{2}$.

Remark: for $B: V \to V^*$ symmetric ($B^* = B$), take $H_B(v) = \frac{1}{2}(Bv, v)$ as a quadratic Hamiltonian on V, ω.
Then the symplectic gradient of H_B wrt ω is:
$X_{H_B}(v) = \xi v$.

As for the eigenvalues of symplectic matrices:

Prop: Let $A \in Sp_{2n}(\mathbb{R})$, then:
1) $\det A = 1$
2) if λ is an eigenvalue of A so is λ^{-1} (and $\bar\lambda, \bar\lambda^{-1}$)
2) the multiplicities of λ, λ^{-1} are the same, and the multiplicities
3) of 1 and -1 eigenvalues are always even.

prf: 1) since $A^*\omega = \omega$, $A^*\omega^n = \omega^n$, so $\det A = 1$ (ω^n is a volume form)
2) since $A^t J A J^{-1} = Id$, $(A^t)^{-1} = (A^{-1})^t = JAJ^{-1}$, so:
$\chi_A(\lambda) = \det(A - \lambda I) = \det((A^{-1})^t - \lambda I) = \det(A^{-1} - \lambda I) = \chi_{A^{-1}}(\lambda) = \chi_A(\lambda^{-1})$.
(note: since A is real matrix $\lambda \in \mathbb{C}$ an eigenvalue $\Rightarrow \bar\lambda \in \mathbb{C}$ an eigenvalue).
3) we leave as exercise (from 1) and 2)) $\square$

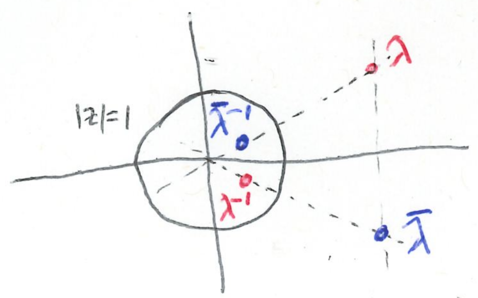
Similarly, for matrices of the Lie algebra:

$\boxed{\text{Prop:}}$ Let $\xi \in \mathfrak{sp}_{2n}(\mathbb{R})$ ($\xi : \mathbb{R}^{2n} \circlearrowleft$, $\xi^t J + J\xi = 0$). Then

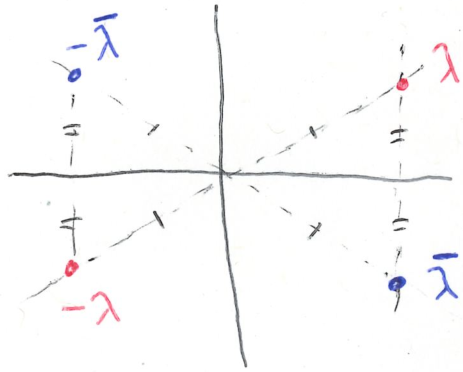
1) if λ is an eigenvalue of ξ, so is $-\lambda$ (and $\bar{\lambda}, -\bar{\lambda}$).

2) the eigenvalue 0 has even multiplicity.

So, we have the following pictures for eigenvalues of symplectic matrices $A \in Sp_{2n}(\mathbb{R})$ or 'infinitesimal' symplectic matrices $\xi \in \mathfrak{sp}_{2n}(\mathbb{R})$:



$Av = \lambda v$ ($A \in Sp_{2n}(\mathbb{R})$)



$\xi v = \lambda v$ ($\xi \in \mathfrak{sp}_{2n}(\mathbb{R})$).

$\boxed{\text{Some topology}}$

$\boxed{\text{Prop:}}$ $\Lambda(n) \approx U_n / O_n$ $\left(\Lambda^+(n) \approx U_n / SO_n\right)$,

for U_n the unitary group ($A : \mathbb{C}^n \circlearrowleft$ $n \times n$ cplx matrices w/ $\langle Az, Aw \rangle = \langle z, w \rangle$ $\forall z, w \in \mathbb{C}^n$)

hermitian product

and O_n the orthogonal group ($R : \mathbb{R}^n \circlearrowleft$ $n \times n$ real matrices w/ $Ru \cdot Rv = u \cdot v$)

(we embed $O_n \hookrightarrow U_n$ by $R(u + iv) := Ru + iRv$).

prf: we consider the standard structures:

$$\mathbb{R}^{2n}, \omega = dp \wedge dq \longleftrightarrow \mathbb{C}^n, \langle z, w \rangle = z \cdot \bar{w} + i\,\omega(z, w)$$

then, any $U \in U_n$ is also a linear symplectic transformation:

$\langle Uz, Uw \rangle = \langle z, w \rangle \Rightarrow \omega(Uz, Uw) = \omega(z, w)$ by taking the imaginary part.

So we can consider $U_n \subset Sp_{2n}(\mathbb{R})$ as a subgroup.

(explicitly, if we write $U = X + iY \in U_n$ in its real/imaginary parts, then

$$U = X + iY \longleftrightarrow \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in Sp_{2n}(\mathbb{R})).$$

Linear symplectic transformations send Lagr. subspaces to Lagr. subspaces so we have an action:

$$U_n \circlearrowright \Lambda(n) \qquad (\text{or } \Lambda^+(n)).$$

This action is transitive: let $e_1, \dots, e_n \in \mathbb{C}^n$ standard unitary basis ($\langle e_j, e_k \rangle = \delta_{jk}$) and $L \subset \mathbb{C}^n$ a Lagrangian subspace. (as a real v.s. $\dim_{\mathbb{R}} L = n$) Take $E_1, \dots, E_n \in L$ an orthonormal basis ($E_j \cdot E_k = \delta_{jk}$), then this is also a unitary basis:

$$\langle E_j, E_k \rangle = E_j \cdot E_k + i \underbrace{\omega(E_j, E_k)}_{0 \text{ since } L \text{ is Lagr.}} = \delta_{jk}.$$

Then $Ue_j = E_j$ is a unitary transf. of $\mathbb{C}^n, \langle \cdot, \cdot \rangle$ and

$$U(L_0) = L \qquad (\text{for } L_0 = \mathrm{span}_{\mathbb{R}}(e_j) = \mathbb{R}^n \subset \mathbb{C}^n).$$

The stabilizer of L_0 are the unitary matrices sending $\mathrm{span}_{\mathbb{R}}\{e_j\} \to \mathrm{span}_{\mathbb{R}}\{e_j\}$, ie O_n, so that $\Lambda(n) \simeq U_n / O_n$. ☐

(note as $2n \times 2n$ sympl. matrices $R \in O_n$ corresponds to $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \in Sp_{2n}(\mathbb{R})$)

Remark: the fibration $U_n \xrightarrow{O_n} \Lambda(n)$ with O_n fibers can be used to compute $\pi_1(\Lambda(n)) \cong \mathbb{Z}$.

Prop: $Sp_{2n}(\mathbb{R}) \approx \mathbb{R}^{n(n+1)} \times U_n \approx \mathbb{R}^{n(n+1)} \times SU_n \times S^1$.

prf: for $A \in Sp_{2n}(\mathbb{R})$ write

$$A = PQ, \qquad Q \in O_{2n}, \; P \text{ symmetric, positive definite with } P^2 = AA^t.$$

we first claim that $P \in Sp_{2n}(\mathbb{R})$ is symplectic.

Indeed, AA^t is symplectic; (A symplectic $\Rightarrow A^t$ symplectic:

$A^t J A = J \Rightarrow J^{-1} = A^{-1} J^{-1} A^{-t} \Rightarrow J = A^{-1} J A^{-t} \Rightarrow A J A^t = J$),

so if we let $a_1, \dots, a_{2n} \in \mathbb{R}_{>0}$ the eigenvalues of AA^t with eigenvectors $e_1, \dots, e_{2n}$ we have:

$\omega(e_j, e_k) = a_j a_k \, \omega(e_j, e_k)$

so that if $a_j a_k \neq 1$, $\omega(e_j, e_k) = 0$. Now $Pe_j = \sqrt{a_j}\, e_j$,

so that if $a_j a_k = 1$ we have $\omega(e_j, e_k) = \omega(Pe_j, Pe_k)$ ($\sqrt{a_j a_k} = 1$)

whereas if $a_j a_k \neq 1$ then $\omega(Pe_j, Pe_k) = \sqrt{a_j a_k}\, \omega(e_j, e_k) = 0 = \omega(e_j, e_k)$.

So $Q = P^{-1} A \in O_{2n} \cap Sp_{2n}(\mathbb{R})$ is also symplectic, and thus unitary ($QQ^t = \mathrm{id}$ & $Q^t J Q = J \Rightarrow JQ = QJ$ (unitary)).

Writing $P = e^S$ for $Se_j = (\frac{1}{2}\log a_j) e_j$ symmetric, we have $S \in Sym_{2n}(\mathbb{R}) \cap \mathfrak{sp}_{2n}(\mathbb{R})$ ($P^{-1}\dot{P} = \dot{S} \in \mathfrak{sp}_{2n}(\mathbb{R})$)

ie $S = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ for $x^t = x$, $y^t = y$ two $n \times n$ symmetric matrices

(ie $2\left(\frac{n(n+1)}{2}\right) = n(n+1)$ real parameters in $\mathbb{R}^{n(n+1)}$). In summary,

send $A \longleftrightarrow (x, y, Q)$ $\quad x^t = x,\ y^t = y,\ Q \in U_n$ $\quad (x,y) \longleftrightarrow \mathbb{R}^{n(n+1)}$ ($x^t = x$, $y^t = y$)

$\in \mathbb{R}^{n(n+1)} \times U_n \qquad e^{\begin{pmatrix} x & y \\ y & -x \end{pmatrix}} Q = A$. $\square$

$\boxed{\text{Remark}}$: One can show SU_n is simply connected by considering it acts transitively ($n > 1$) on the sphere $S^{2n-1} \circlearrowleft SU_n$ with stabilizers SU_{n-1}. Using induction and the fibrations

$SU_n \xrightarrow{SU_{n-1}} SU_n / SU_{n-1} = S^{2n-1}$ (with SU_{n-1} fibers).

In particular, one obtains that $\pi_1(Sp_{2n}(\mathbb{R})) \cong \mathbb{Z}$.

Remark: To any *loop* $t \mapsto A(t) = A(t+1) \in Sp_{2n}(\mathbb{R})$ of symplectic matrices, one can associate (in a certain way to take account of sign) an integer:

$$\mu_+(A) = [A(t)] \in \pi_1(Sp_{2n}(\mathbb{R})) \simeq \mathbb{Z} \quad \text{(the MASLOV INDEX of the loop)}$$

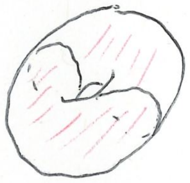
And likewise to any *loop* $t \mapsto L(t) = L(t+1) \in \Lambda(n)$ of Lagrangian subspaces an integer (also called the loops' maslov index) $\mu(L) = [L(t)] \in \pi_1(\Lambda(n)) \simeq \mathbb{Z}$.

Example: For $n=1$ $(\mathbb{R}^2, dx \wedge dy)$, we have:

$$\Lambda(1) = \mathbb{RP}^1 \simeq S^1, \quad Sp_2(\mathbb{R}) = SL_2(\mathbb{R}) \simeq \mathbb{R}^2 \times S^1 \simeq D^\circ \times S^1$$

$$D^\circ = \{z \in \mathbb{R}^2 : |z| < 1\}$$

$\Lambda(1)$



$Sp_2(\mathbb{R}) = SL_2(\mathbb{R})$ is the *interior* of a solid torus.

To place some 'landmarks' in this solid torus, consider the eigenvalues of $A \in Sp_2(\mathbb{R})$ are roots of

$$\lambda^2 - tr(A)\lambda + 1 = 0$$

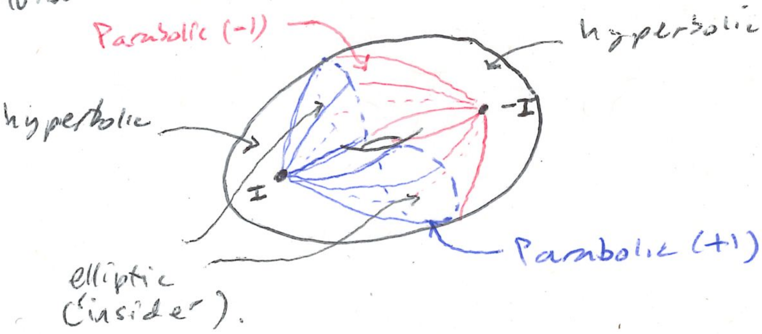
and we have the following types:

1) $|trA| < 2$ $(\lambda, \bar{\lambda} \in S^1 \setminus \{1, -1\})$, Elliptic

2) $|trA| = 2$ $(\lambda_1 = \lambda_2 = \pm 1)$, Parabolic

3) $|trA| > 2$ $(\lambda_1 = \lambda_2^{-1} \in \mathbb{R} \setminus \{0, \pm 1\})$, Hyperbolic

In the torus they look like: ($Sp_2(\mathbb{R}) \setminus \{\text{Parabolic}\}$ has 4 connected components)

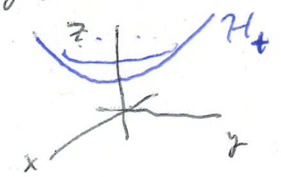


which one can check by coordinatizing for example by

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in Sp_2(\mathbb{R}), \quad ac - b^2 = 1, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$$

and taking $a = z + x$, $c = z - x$, $b = y$ for $(x,y,z) \in \mathcal{H}_+ \subset \mathbb{R}^3$ in the upper sheet of a 2-sheeted hyperboloid:

$$z^2 - x^2 - y^2 = 1 \quad (z > 0)$$



or $z = \cosh\psi$, $x = \sinh\psi\cos\varphi$, $y = \sinh\psi\sin\varphi$,

or $r = \tanh\psi \in [0,1)$, $\varphi, \theta \in \mathbb{R}/2\pi\mathbb{Z}$ $\left((x,y,z) = \frac{(r\cos\varphi, r\sin\varphi, 1)}{\sqrt{1-r^2}} \in \mathcal{H}_+ \right)$

to identify $Sp_2(\mathbb{R}) \simeq D^\circ \times S^1 \ni (r\cos\varphi, r\sin\varphi, \theta)$.

In these coordinates: $tr(A) = (a+c)\cos\theta = 2z\cos\theta = \dfrac{2\cos\theta}{\sqrt{1-r^2}}$,

and the parabolic matrices $((tr A)^2 = 4)$ are the locus:

$$r^2 = \sin^2\theta.$$

$\boxed{\text{Remark:}}$ In practice it is not usually loops of matrices (or loops of Lagr. subspaces) that arise but rather <u>paths</u> of them by linearization of a flow along an trajectory:

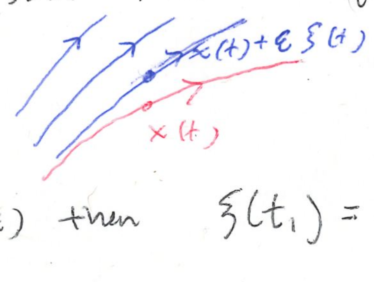
consider a (possibly time dependent) system of ODE's:

$$(*) \quad \dot{x} = X(x,t) \quad (x \in \mathbb{R}^m)$$

and suppose $x(t)$ is a given solution, and write $\phi_{t_0,t_1} : \mathbb{R}^m \circlearrowleft$ for the 'flow' from time $t_0 \to$ time t_1 of $(*)$ (so $\phi_{t_0,t_1}(x(t_0)) = x(t_1)$). The linearized eq's along the solution $x(t)$ are then the system:

$$(**) \quad \dot{\xi} = \Xi(t)\cdot\xi, \qquad \Xi(t) = d_{x(t)}X|_t$$

(if $x(t) + \varepsilon\xi(t)$ is a sol'n then sending $\varepsilon \to 0$, ξ satisfies $(**)$)



and if $\xi(t)$ solves $(**)$ then $\xi(t_1) = (d_{x(t_0)}\phi_{t_0,t_1})\xi(t_0)$.

In particular, if we consider for example a Hamiltonian v.f.

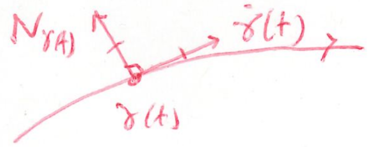
$$\dot{x} = X_H(x) \quad , \quad x \in \mathbb{R}^{2n} ,$$

then $(**)$ has a 'path' $t \longmapsto \Xi(t) \in sp_{2n}(\mathbb{R})$,

and $t \longmapsto (d_{x(t_0)} \Phi_{t_0,t}) \in Sp_{2n}(\mathbb{R})$ is a path of symplectic matrices.

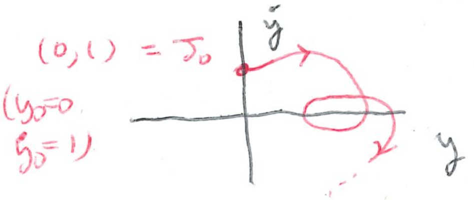
$\boxed{\text{Example:}}$ let $t \mapsto \gamma(t)$ be a unit speed geodesic on a surface (or more generally in (M,g)). The linearization of the unitspeed geodesic flow along $\gamma(t)$ is described by the normal Jacobi fields:

$$J(t) = y(t) N_{\gamma(t)}$$

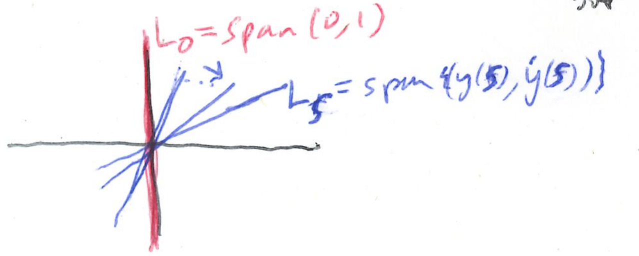


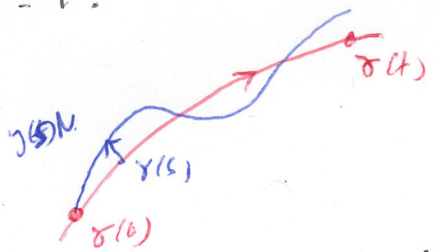
$$\ddot{y} = -K(t) y \quad , \quad K(t) = K(\gamma(t)) \text{ Gaussian curvature of the surface } @\gamma(t).$$

We can draw such Jacobi fields as curves in the $(y, \dot{y})$ plane,



The # of conjugate points between $\gamma(0)$ and $\gamma(t)$ is the total # of 'half turns' of the line $\mathrm{span}(y(s), \dot{y}(s)) = L_s$: for $0 \le s \le t$.





or, in the above language, the linearized geodesic flow along the geodesic $t \mapsto \gamma(t)$ has an associated path $t \longmapsto L_t \in \Lambda(1) = \mathbb{RP}^1$ of Lagrangian subspaces (lines)

the "Maslov index of this path" (one needs to define a consistent way to 'close' a path into a loop to talk about index of these paths) ~~for the~~ from $t = 0$, to $t = t_1$ is the # of conjugate points along γ between $\gamma(0)$ and $\gamma(t_1)$.

Compatible complex structures

We have used above the identification $\mathbb{R}^{2n} \longleftrightarrow \mathbb{C}^n$, based upon $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, $J^2 = -I$ as our standard complex structure on $\mathbb{R}^{2n}$.

More generally:

$\boxed{\text{Def:}}$ A complex structure on a vector space V is a linear map $J : V \circlearrowleft$ s.t. $J^2 = -I$.

$\boxed{\text{Exercise:}}$ If V admits a complex structure J then $\dim V = 2n$ is even (take determinants of $J^2 = -I$). Given a cplx str. (V, J) we have on V ($\dim_{\mathbb{R}} V = 2n$) as well the structure of an n-dimensional complex vector space by $(a+ib)v := av + bJv$ $(a+ib \in \mathbb{C})$.

$\boxed{\text{Def:}}$ For (V, ω) a ~~complex~~ symplectic vector space, a complex structure J on V is called ω-compatible if

1) $J^* \omega = \omega$ 2) $\omega(Jv, v) > 0$ $\forall v \in V$.

we write

$$\mathcal{J}(V, \omega)$$

for all the ω-compatible complex structures on V, ω.

$\boxed{\text{Exercise:}}$ Check that given $J \in \mathcal{J}(V, \omega)$ then

$$g_J(u, v) := \omega(Ju, v)$$

defines a positive definite inner product on V, and that

$$\langle u, v \rangle_J := g_J(u, v) + i\,\omega(u, v)$$

a Hermitian product on V considered as an n-dim. cplx. vector space.

Moreover check that:

$$J^* g_J = g_J, \quad \omega(u, v) = g(u, Jv).$$

Prop: 1) There is a projection:

$$\mathrm{Sym}^2_+(V) \longrightarrow \mathcal{J}(V,\omega)$$

positive definite, symmetric, bilinear forms on V (inner products).

2) $$\mathcal{J}(V,\omega) \approx \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathrm{Sym}^2_+(\mathbb{R}^n)$$

(in particular $\dim \mathcal{J}(V,\omega) = n(n+1)$, and it is a path connected (contractible) space).

prf: for (1), let $g \in \mathrm{Sym}^2_+(V)$ and write

$$\omega(u,v) = g(u, Kv), \quad K : V \to V \; [K^t = -K \text{ since } \omega \text{ is skew}].$$

(if $K^2 = -I$, we are done) otherwise take the decomposition:

$$K = P\alpha$$

$\alpha \in O(V,g)$ ($\alpha^t\alpha = I$) and P is positive def., symmetric w/ $P^2 = KK^t = -K^2$. Then we check (consider eg an eigenspace of K)

$$[PK = KP], \text{ ie } KP^{-1} = P^{-1}K.$$

Now, we claim that $\alpha = J \in \mathcal{J}(V,\omega)$ is an ω-compatible cplx. str.

Indeed: $\alpha^2 = P^{-1}KP^{-1}K = P^{-2}K^2 = -I$,

so that α is a cplx. str. on V. And for ω-compatibility, we have

$$K^t = -K = \alpha^t P = -\alpha P \Rightarrow [K = \alpha P = P\alpha],$$

so that $\omega(\alpha u, \alpha v) = g(\alpha u, \alpha P \alpha v) = g(u, P\alpha v) = \omega(u,v)$.

ie $\alpha^*\omega = \omega$ so $\alpha = J \in \mathcal{J}(V,\omega)$ ✓.

For (2), we can fix some 'base' Lagrangian subspace $L_0 \subset V$ (and fix a complementary $L_0' : V = L_0 \oplus L_0'$).

Let $J \in \mathcal{J}(V,\omega)$. Then $JL_0 \in \Lambda(n)$ is also Lagrangian, and moreover it is <u>transverse</u> to L_0: if $v \in L_0 \cap JL_0$ then $g_J(v,v) = \omega(Jv,v) = 0$ ($Jv, v \in L_0$ which is Lagrangian), but g_J is positive definite, so we must have $v=0$ (ω-comp. of J).

So any $J \in \mathcal{J}(V,\omega)$ has associated $JL_0 \pitchfork L_0$, which as we have seen can be parametrized by symmetric $n\times n$ matrices (in corresponding $\dim \Lambda(n) = \frac{n(n+1)}{2}$). Suppose next that $JL_0 = J'L_0 = L \pitchfork L_0$ for some $J, J' \in \mathcal{J}(V,\omega)$. Then from the identification $V = L_0 \oplus L \cong L_0 \times L_0^*$, we have for $u, v \in L_0$:

$$u + Jv \mapsto (u, b(v)),\ (u, b'(v'))$$

where $v' = -J'Jv$, and

$$b : L_0 \to L_0^*,\ \ell \mapsto g_J(\ell,\cdot),\quad b' : L_0 \to L_0^*,\ \ell \mapsto g_{J'}(\ell,\cdot)$$

present the restrictions $g_J|_{L_0}$, $g_{J'}|_{L_0}$.

Since $J : L_0 \to L$, $J' : L_0 \to L$ are isometries of g_J, $g_{J'}$ the formula $b(v) = b'(-J'Jv)$ reads:

$$b'J' = bJ$$

so that, when $JL_0 = J'L_0$, we have $J = J'$ iff $b' = b$, ie $g_J|_{L_0} = g_{J'}|_{L_0}$ which is a positive definite symmetric $n\times n$ matrix (ie in summary, the map

$$\mathcal{J}(V,\omega) \ni J \longmapsto (JL_0,\ g_J|_{L_0}) \in \Lambda^0(n) \times \mathrm{Sym}^2_+(L_0)$$

is our diffeo ($\Lambda^0(n)$ the Lagr. subspaces transverse to L_0). $\square$

Remark: we can see path connectedness just from the projection (1): Let $J_0, J_1 \in \mathcal{J}(V,\omega)$ and consider $g_t = (1-t)g_{J_0} + t\, g_{J_1} \in \mathrm{Sym}^2_+(V)$ which project under (1) to a path $J_t \in \mathcal{J}(V,\omega)$ from J_0 to J_1.

Remark: We have similar properties for symplectic vector bundles.

Let $\mathbb{R}^{2n} \longrightarrow E \longrightarrow B$

a (real) vector bundle of rank $2n$ over B ($\dim_{\mathbb{R}} E_b = 2n$).

We call a complex vector bundle structure J on E a (smooth) collection of complex structures $J_b : E_b \circlearrowleft$, $J_b^2 = -I$, and a symplectic vector bundle structure ω on E a (smooth) collection of symplectic structures, $\omega_b \in \Lambda^2(E_b^*)$ on the fibers $E_b = \pi^{-1}(b)$.

A complex vector bundle structure J on E is compatible with a symplectic vector bundle structure (E,ω) when it is on each fiber: $J_b \in \mathcal{J}(E_b, \omega_b)$, and we write $J \in \mathcal{J}(E,\omega)$.

Then any symplectic vector bundle (E,ω) admits a compatible complex vector bundle structure $J \in \mathcal{J}(E,\omega)$: the same partition of unity argument as on Riem. mflds. gives the existence of an inner product structure $g_b : E_b \times E_b \to \mathbb{R}$, g, on E. Apply (1) of the last proposition to obtain on each (E_b, ω_b, g_b) a compatible (E_b, ω_b, J_b), ie a $J \in \mathcal{J}(E,\omega)$.

This space $\mathcal{J}(E,\omega)$ is still path connected, by the same argument of the last remark.

In particular, any symplectic manifold (M,ω), we may view $TM \longrightarrow M$ as a symplectic vector bundle (T_xM, ω_x) and so always have existence of a compatible complex vector bundle structure (TM, J) on TM ($J \in \mathcal{J}(M,\omega)$). Such a J is called an almost complex structure on M (it need not correspond to an atlas on M, $\mathbb{C}^n$ valued with holomorphic transition functions).