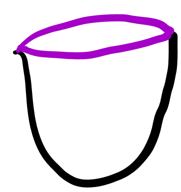
Applications of Symplectic/Floer theory: (mostly, to the 3BP) Class of 27/01/25

So far, we have seen two floer theories:

	For periodic orbits	For Lagr. intersections
On (M,ω) closed	Ham. Floer theory	Lagrangian Floer theory

What if (M,ω) has boundary?

- too general
- but say $\omega = d\lambda$, and ∂M is of restricted contact-type ($\Rightarrow M$ is a Liouville domain)



Then, we can define:

	For periodic orbits	For Lagr. intersections
On a Liouville domain	Symplectic homology	Wrapped Floer homology

Take-away: there are many variants of Floer theory, depending on the problem we study.

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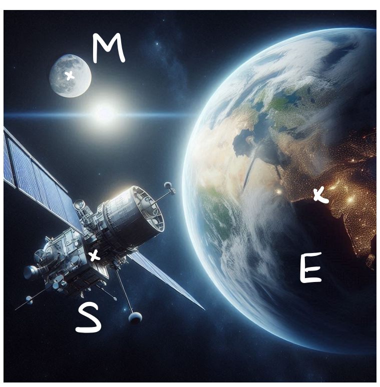
eg of interest:

Circular Restricted 3-Body Problem

Set-up: 3 masses (Earth, Moon, ~~Satellite~~) under the influence of Newtonian gravity.

$$\left(|F_{ab}| = \frac{G m_a m_b}{|q_a - q_b|^2} \right)$$



Assumptions: • $m_S = 0$ (RESTRICTED)

• M moves in a circle around E (CIRCULAR)

• Note: IRL, the Moon's orbit has eccentricity $e \approx 0.05$

Thm (Moreno, van Koert)[2020]: for low energies, can reduce this problem to a Hamiltonian problem on $W \simeq \mathbb{D}^* S^2$.

3 of position ⊕ 3 of momentum

(So drop from 6 dimensions to 4. Will be more precise later).

symplectic homology

Cor. $SH_*(\mathbb{D}^* S^2)$ determines periodic orbits in the CR3BP.

Cor. $HW_*(\mathbb{D}^* S^2)$ determines Lagrangian intersections in the CR3BP.

what are these? → trajectories with specific boundary conditions.

↳ see Moreno – L. 2024 (both papers)

Example of application:

Thm (Moreno, van Koert 2021)

$\exists\ \infty^{ly}$ many periodic orbits in the CR3BP.

<u>Pf sketch:</u>

- can reduce the problem to $\mathbb{D}^*S^2$
- a famous theorem (Abbondandolo-Schwarz 2004) shows that $\forall$ manifold M:

off-the-shelf

$$SH_*(\mathbb{D}^*M) \cong H_*(\Omega M)$$

singular homology

based loop space on M

- standard algebraic topology shows that

$$\dim H_*(\Omega S^2) = \infty$$

• $\Rightarrow \dim SH_*(\mathbb{D}^* S^2) = \infty$

• but $SH_*(\mathbb{D}^* S^2)$ counts periodic orbits (up to homology).

• $\therefore \exists \infty^{ly}$ many orbits. $\square$

• Note: This is very simplified. In real life, one needs to be very careful, especially since our Hamiltonian could be degenerate.

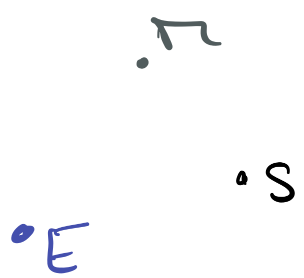
Also need more assumptions (twist condition, convexity range ...)

See [Moreno, van Koert 2021] for full proof.

II. Reducing problems to Floer problems.

- if already working on a closed symp. manifold/Liouville domain: ✓
- Else, how do we produce one?

• eg: CR3BP



- $m_S = 0$
- M moves and E in a circle

($\Rightarrow$ can choose a rotating frame in the Earth-Moon plane $\Rightarrow$ fix their pos. to $\vec{M}$ and $\vec{E}$)

Want to study the satellite.

Position space: $\mathbb{R}^3$

Phase space: $T^*\mathbb{R}^3 \cong \mathbb{R}^6 = \{(q,p)\}$

Hamiltonian of the satellite:

$$H : T^*\left(\mathbb{R}^3 \setminus \{\vec{E}, \vec{M}\}\right) \longrightarrow \mathbb{R}$$

$$(q,p) \longmapsto \frac{1}{2}|p|^2 - \frac{m_E}{|q-\vec{E}|} - \frac{m_M}{|q-\vec{M}|} \underbrace{+ q_1 p_2 - q_2 p_1}$$

$\boxed{\text{angular momentum}}$

This term appears bc we chose a rotating frame in the $q_1 q_2$-plane.

→ This is a Hamiltonian on $\mathbb{R}^6$.

Pbm: it has singularities.

Indeed, $q \to \vec{E}$, or $q \to \vec{M}$ (= collision)

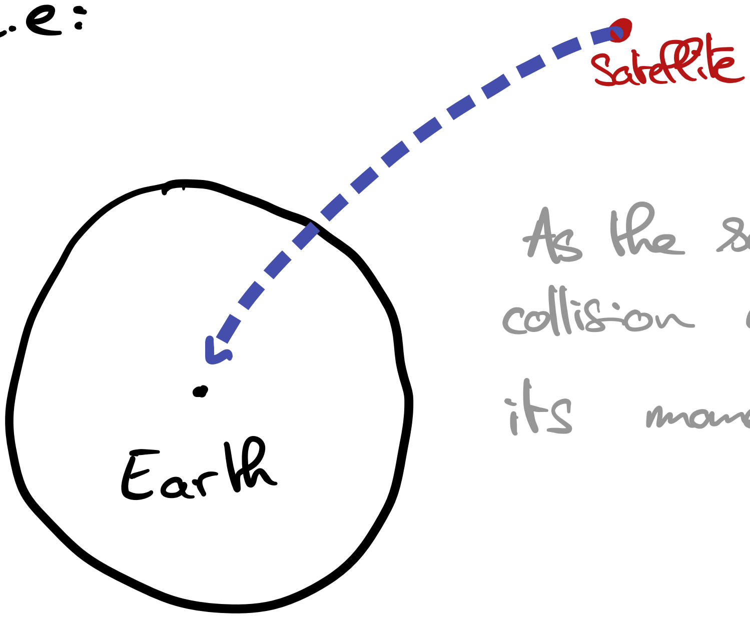
⇒ one of the terms $\frac{m_E}{|q-\vec{E}|}$ or $\frac{m_M}{|q-\vec{M}|}$ blows up to ∞

But recall that $H \equiv cst$ along flow lines (conservation of energy) so

$$H(q,p) = \underbrace{\frac{1}{2}|p|^2}_{\text{must go to } \infty \text{ too.}} - \underbrace{\frac{m_E}{|q-\vec{E}|}}_{\to \infty} - \underbrace{\frac{m_M}{|q-\vec{M}|}}_{\text{finite}} + \underbrace{q_1 p_2 - q_2 p_1}_{\text{finite}}$$

So $(q \to \vec{E} \text{ or } \vec{M}) \Rightarrow p \to \infty$

i.e:



As the satellite approaches collision with the Earth, its momentum goes to ∞.

$\therefore$ Energy hypersurfaces $H^{-1}(c)$

are <u>non-compact.</u>

↳ how to compactify them?

<u>Regularization at collisions.</u>

<u>Regularizing</u> (in this context) means

changing coordinates so as to compactify

$H^{-1}(c)$.

→ by now standard. Many ways to do this, depending on what we want our regularization to satisfy.

Moser regularization

Right now we have:

$$(q \to \vec{E} \text{ or } \vec{M}) \Rightarrow (p \to \infty)$$

So $H^{-1}(c) \subseteq T^*(\mathbb{R}^3 \setminus \{\vec{E}, \vec{M}\})$

and the blow up happens in the fibers as q approaches $\vec{E}$ or $\vec{M}$ in the base space.

⚠ Pblm: blow up in the fibers? Not very convenient.

Would be easier to compactify base space!

Recipe:

$$T^*\mathbb{R}^3 \xrightarrow{\text{swap}} T^*\mathbb{R}^3 \xrightarrow[\text{p-space}]{\text{compactify}} T^*S^3$$

$$(q,p) \longmapsto (p,-q)$$

so blow up now happens in the base copy of $\mathbb{R}^3$ (in $T^*\mathbb{R}^3 = \mathbb{R}^3 \oplus \mathbb{R}^3$)

The reason for the − sign is so that the symp. structure is preserved

by adding the point $\{p=\infty\}$. Formally, we simply apply an inverse stereographic proj.

New Hamiltonian in these coordinates:

$$\tilde{H} : T^*S^3 \longrightarrow \mathbb{R}$$ (regularized Hamiltonian)

Conclusion: after regularizing at collisions, CR3BP is described by:

$$\tilde{H} : T^*S^3 \to \mathbb{R}$$

$\rightsquigarrow$ by conservation of energy, motion of the satellite is constrained to

$$\tilde{H}^{-1}(c) \cong S^*S^3 \quad - \text{ compact!}$$

(not obvious, relies on our expr. for $\tilde{H}$,)

for low energies.

Thm: (Albers, .. 2011; Cho, Jung, Kim 2018) The flow of the satellite on S^*S^3 is a Reeb flow (i.e., it is induced by a contact struct.)

In summary:

have a Reeb flow on S^*S^3 - compact

⚠ **Pblm:** our Floer theories are defined on symp. mfds (even-dimensional)

$$\dim S^*S^3 = 5$$

Can we still drop one dimension?

Yes!

Great insight from Poincaré, in «Poincaré's last geometric theorem» (1912)

in the ctxt of studying the 3BP

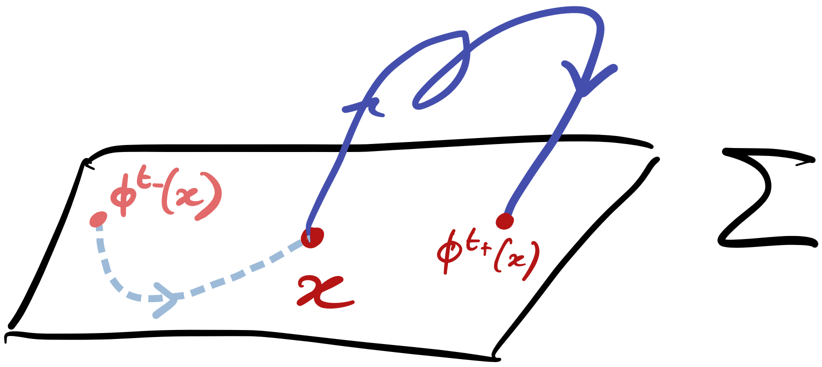
Def^n: Let M any manifold, and $\phi^t : M \longrightarrow M$ a flow.

A hypersurface $\Sigma \hookrightarrow M$ is called a **Poincaré section** if:
(or global hypersurface of section)

- ϕ^t is transverse to $\mathrm{int}(\Sigma)$
- $\partial\Sigma$ is ϕ^t-invariant (if $\neq \emptyset$)
- $\forall x \in \Sigma$: $\exists \begin{cases} t_+ > 0 \\ t_- < 0 \end{cases}$ s.t

$$\underbrace{\phi^{t_\pm}(x) \in \Sigma}$$

ie, $\forall x \in \Sigma$, the flow of x returns to Σ both in the past and future.



Define 1st return map of the flow:

$$\tau : \Sigma \longrightarrow \Sigma$$
$$x \longmapsto \phi^{t_+}(x)$$

(where $t_+ := \inf\{ t_+ > 0 \mid \phi^{t_+}(x) \in \Sigma \}$)

Why this definition?

Poincaré's insight:

$$\underbrace{\left\{ \begin{array}{l} \text{periodic orbits} \\ \text{with base} \\ \text{point in } \Sigma \end{array} \right\}}_{\text{continuous pbm in } n \text{ dimensions}} \longleftrightarrow \underbrace{\left\{ \begin{array}{l} \text{fixed points} \\ \text{of } \tau : \Sigma \to \Sigma \end{array} \right\}}_{\text{discrete pbm in } n \text{ dimensions}}$$

Thm: in the CR3BP, after regularizing at collisions and restricting to an energy level set $H^{-1}(c)$,

$\exists$ Poincaré section $W \hookrightarrow H^{-1}(c)$.

More precisely:

	Planar CR3BP	Spatial CR3BP
Proven by	Poincaré (1912)	Moreno, van Koert (2020)
Phase space	$T^*\mathbb{R}^2$	$T^*\mathbb{R}^3$
Regularized phase space	T^*S^2	T^*S^3
$H^{-1}(c)$	$S^*S^2 \cong \mathbb{RP}^3$	$S^*S^3 \cong S^3 \times S^2$
W	$\mathbb{D}^*S^1 \simeq$ std. annulus	$\mathbb{D}^*S^2$

Liouville domains.
(proven in problem sheet 9)

Can now construct Floer theories on these Poincaré sections, since they are Liouville domains.

III. Concrete applications of Floer theory \ to engineering pbms

Floer theory provides groups HF^* to describe physical pbms globally.

considers all orbits/Lagr. intersections at once

Hence well-suited for statements like

« $\exists\ \infty$ ly many orbits/intersections »

which make use of the global topology of the underlying manifolds.

→ Can also do this locally!

(Oh 96, Ginzburg 10, Moreno-L. 24)

Given one orbit  x

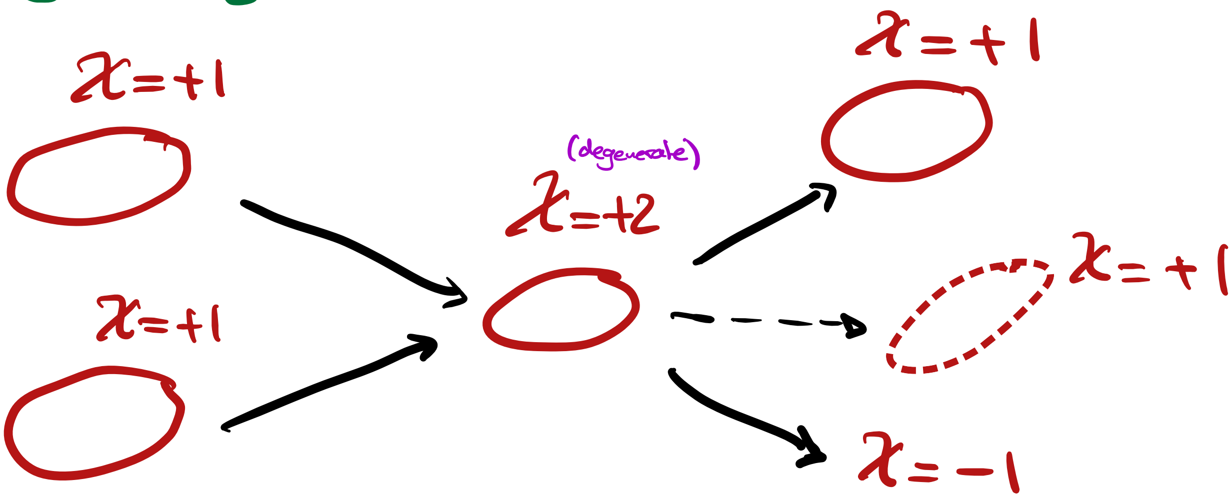
Can extract an integer invariant $\chi \in \mathbb{Z}$ from Floer homology. Numerically computable (Aydin 23, Aydin, Moreno, van Koert, 24...)

s.t.:

- if x is non-degenerate, $\chi(x) = \pm 1$
- $\chi(x)$ is <u>perturbation-invariant</u>

<u>Pfs</u>: advanced Floer homology.

<u>eg.</u> Bifurcation



So, if we do numerical continuation of trajectories, this invariant tells us whether we have missed orbits.

State-of-the-art:

The methods discussed today are very popular in 3 research groups:

and all their subsequent students

- Urs Frauenfelder (Augsburg) → general
- Otto van Koert (Seoul, Korea) → focus on numerics
- Agustin Moreno (Heidelberg) → focus on dynamics (Anosov flows)

all interested in Floer theory