Morse homology

Let M be a mfd. A function $f \in C^\infty(M, \mathbb{R})$ is called Morse if all critical points are nondegenerate. This means the following:

$x \in M$ is critical if $d_x f = 0$.

$x \in Crit(f)$ is nondegenerate if the Hessian of f at x is invertible.

(This notion can be defined in local coordinates and the definition does not depend on the choice of local coordinates.)

Lemma [Morse]

$p \in Crit(f)$. Then, $\exists$ coordinate chart $\varphi : \underset{p}{\underset{\psi}{U}} \longrightarrow \mathbb{R}^n$ such that

$$f \circ \varphi^{-1}(x_1, \dots, x_n) = f(p) - (x_1^2 + \dots + x_k^2) + (x_{k+1}^2 + \dots + x_n^2) \quad \text{for some}$$

$0 \le k \le n$.

Proof: Moser's trick.

The integer k in the lemma is called the index of p.

Corollary. Critical points are isolated. If M is cpt,

$$\# Crit(f) < \infty.$$

~~Morse~~ functions

Examples. (i) Height function: ~~Take a surface and embed it in $\mathbb{R}^3$.~~

For instance S^2 or $T^2 \subseteq \mathbb{R}^3$. The restriction of $(x,y,z) \mapsto z$ to S^2 or T^2 is Morse.

On S^2, it has 2 critical points, the max and the min. (II)

On T^2, it has 4, the minimum, a maximum and two saddle points

(ii) On $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, the function $f(x,y) \doteq \cos 2\pi x + \cos 2\pi y$. is Morse with 4 critical points

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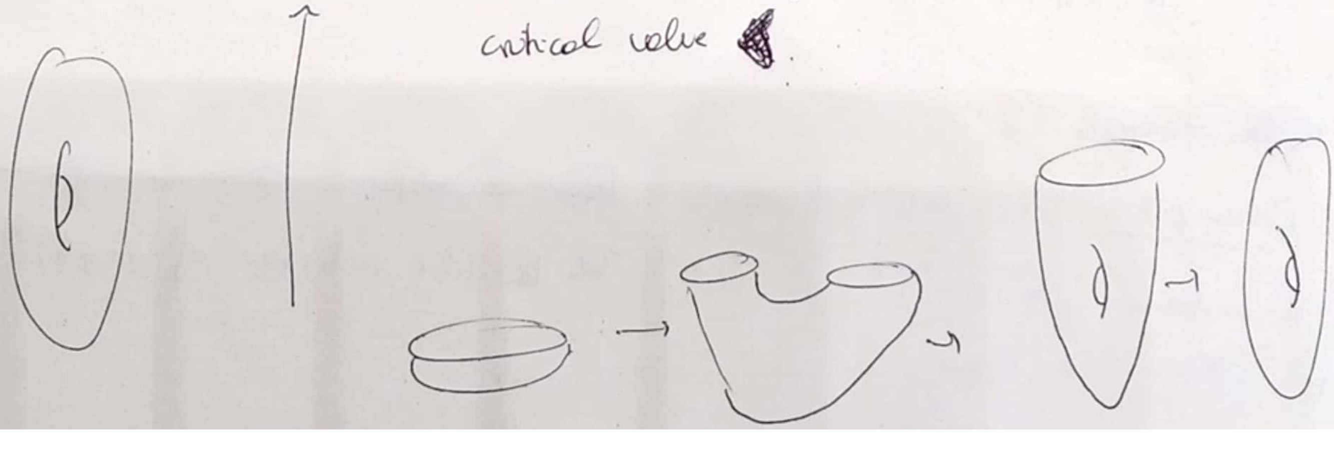
Morse functions encode the topology of manifolds. Consider the torus and the height function. It is easy to see that the topology of sublevel sets changes when we cross a critical value.



Thm ~~Let p~~ Let $f: M \to \mathbb{R}$ be a Morse function.

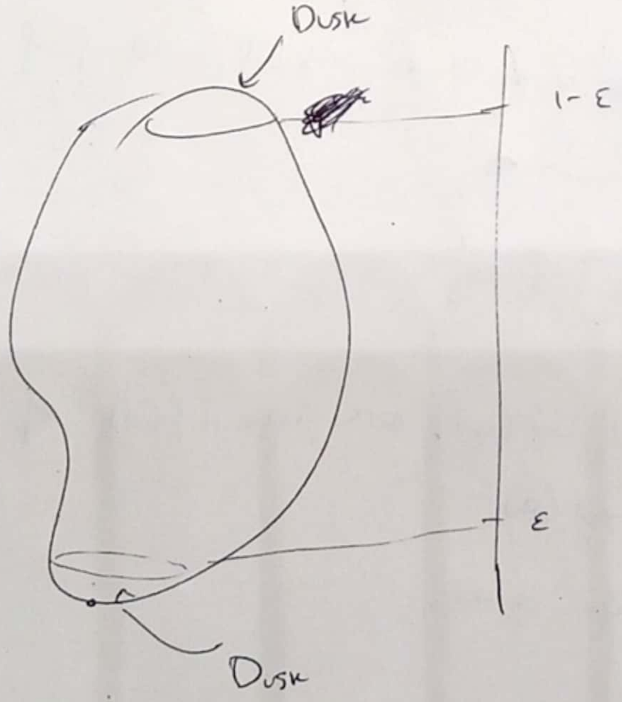
(i) Suppose $a, b \in \mathbb{R}$ are such that $f^{-1}[a,b]$ is compact and $[a,b]$ does not contain any critical value of f. ~~Then~~ ~~Then let~~ for $c \in \mathbb{R}$, let $V^c := f^{-1}(-\infty, c]$. Then V^b deformation retracts onto V^a.

(ii) Suppose $\alpha = f(p)$ is critical value and $\varepsilon > 0$ is such that $f^{-1}[\alpha-\varepsilon, \alpha+\varepsilon]$ is cpt and does not contain any critical value other than α. Then, the homotopy type of $V^{\alpha+\varepsilon}$ is that of $V^{\alpha-\varepsilon}$ with k-cell attached, where $k = \mathrm{ind}(p)$ is the Morse index of p.

Proof see [Audin-Damian].

Corollary (Reeb)
Let M be a cpt mfd and $f: M \to \mathbb{R}$ a Morse function with exactly two critical points. Then M is homeomorphic to a sphere.

Proof: The critical points must be the maximum and minimum.
$\to$ nbds of them must be disks by the Morse lemma.



$V^{1-\varepsilon} \cong V^{\varepsilon}$ by (i) of the above thm
$\Rightarrow V$ is ~~diffeomorphic~~ the union two disks glued along their boundary $\Rightarrow$ homeomorphic to S^n.

Morse functions exist in abundance

Thm Morse functions on (cpt) mflds are dense.

Corollary Manifolds admits cw decompositions.

The last corollary suggests that Morse functions are somewhat related to cellular homology.

Let us see how.

Def A pseudogradient vector field adapted to f is a vector field X such that

(i) $df(X) \leq 0$, where equality holds only at critical points.

(ii) In a Morse chart, $X = -\text{grad} f$. (w.r.t. the canonical metric on $\mathbb{R}^n$)

Remark f is nonincreasing along trajectories of the flow of X.

Exercise Prove that given a Morse function f, there exists a pseudogradient v.f. X adapted to X.

Let φ^s be the flow of X and $a \in M$ a critical point of f.

Define
$$\begin{cases} W^s(a) = \{ p \in M \mid \varphi^s(p) \xrightarrow{s \to +\infty} a \} \\ W^u(a) = \{ p \in M \mid \varphi^s(p) \xrightarrow{s \to -\infty} a \} \end{cases}$$

It can be proven that $W^s(a), W^u(a)$ are manifolds with $\dim W^u(a) = \text{codim}\, W^s(a) = \text{ind}(a)$.

Moreover, they are diffeomorphic to open disks.

The point now is that a manifold decomposes into the flow lines of X and these flow lines connect critical points.
(there is something to prove here).

The moduli space of trajectories connecting critical points is going to be helpful in constructing a chain complex.

$a, b \in \mathrm{Crit}(f)$. Define $\mathcal{M}(a,b) = \left\{ x \in M \;\middle|\; \lim_{s\to-\infty} \varphi^s(x) = a,\ \lim_{s\to+\infty} \varphi^s(x) = b \right\}$.

Clearly, $\mathcal{M}(a,b) = W^u(a) \cap W^s(b)$.

__Def__ We call X Morse-Smale if all unstable and stable manifolds are transverse.

__Thm__ Morse-Smale v.f. are generic.

X Morse-Smale $\Rightarrow$ $\mathcal{M}(a,b)$ is a mfd and

$\operatorname{codim} \mathcal{M}(a,b) = \operatorname{codim} W^u(a) + \operatorname{codim} W^s(b)$.

__Remark__ (i) We are interested in unparameterized trajectories. Note that $\mathbb{R}$ acts on $\mathcal{M}(a,b)$ by translation. The action is free and proper $\Rightarrow$ $\mathcal{M}(a,b)/\mathbb{R} = \mathcal{L}(a,b)$ is a mfd and

$\dim \mathcal{L}(a,b) = \dim \mathcal{M}(a,b) - 1$.

(ii) It is easy to identify $\mathcal{L}(a,b)$ with $\mathcal{M}(a,b) \cap f^{-1}(\alpha)$, where $\alpha \in (f(b), f(a))$ (since all trajectories must go through the level set $f^{-1}(\alpha)$).

Since we are working over cpt mflds, $\mathcal{L}(a,b)$ is cpt.

(iii) Suppose $\text{ind}(a) = \text{ind}(b)+1$

Then $\dim \mathcal{M}(a,b) = n - \text{codim}\, W^u(a) - \text{codim}\, W^s(b)$

$$= \dim W^u(a) - \text{codim}\, W^s(b)$$

$$= \text{ind}(a) - \text{ind}(b) = 1.$$

$\Rightarrow$ $\mathcal{L}(a,b)$ is a cpt 0-dimensional mfd $\rightsquigarrow$ finite set of points.

We are now ready to define the Morse chain complex (with $\mathbb{Z}_2$ coefficients for simplicity).

Define $C_k(f,X) := \mathbb{Z}_2 \cdot \text{Crit}_k(f)$,

where $\text{Crit}_k(f) = \{x \in M \mid x \text{ critical for } f,\ \text{ind}(x) = k\}$.

We need a boundary operator $\partial : C_k(f,X) \to C_{k-1}(f,X)$.

If $a \in \text{Crit}_k(f)$, define

$$\partial a = \sum_{b \in \text{Crit}_{k-1}(f)} n(a,b)\, b, \quad \text{where } n(a,b) = \#\mathcal{L}(a,b) \pmod 2$$

With some effort, it can be shown that $\partial^2 = 0$.

$\Rightarrow$ we get a chain complex.

With some more effort, it can be shown that although the chain complex depends on f and X, its homology does not.

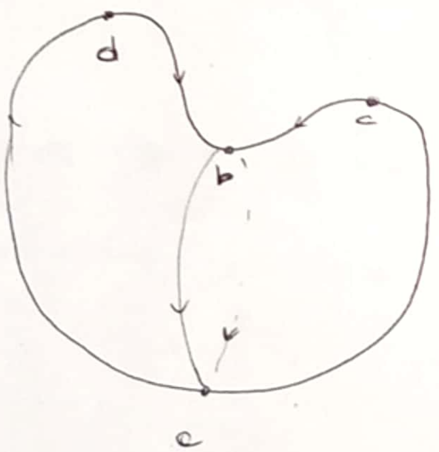
Examples (i) S^2, f = height function.

$$\begin{cases} C_0 = \mathbb{Z}_2 \cdot b \\ C_1 = 0 \\ C_2 = \mathbb{Z}_2\, a \\ \partial = 0 \end{cases}$$

$$\Rightarrow \quad HM_k(S^2) = \begin{cases} \mathbb{Z}_2 & , \ k = 0, 2 \\ 0 & , \ k = 1. \end{cases}$$

(ii) Consider a sphere that looks like this:

f = height function.



Then, $\begin{cases} C_0 = \mathbb{Z}_2 \cdot a \\ C_1 = \mathbb{Z}_2 \cdot b \\ C_2 = \mathbb{Z}_2 \cdot c \oplus \mathbb{Z}_2 \cdot d \end{cases}$

$$\begin{cases} \partial d = \partial c = b \\ \partial b = 2a = 0 \\ \partial a = 0. \end{cases}$$

$$\Rightarrow \begin{cases} HM_0 = \mathbb{Z}_2 & , \text{ generated by } a. \\ HM_1 = 0 \\ HM_2 = \mathbb{Z}_2 & , \text{ generated by } d-c. \end{cases}$$

Thm Morse homology is isomorphic to cellular homology (and hence to singular homology).

Morse inequalities

Thm. M cpt, $f : M \to \mathbb{R}$ Morse function.

Then $\#\mathrm{Crit}(f) \geq$ sum of Betti numbers of M

$$= \sum_{i=0}^{n} \mathrm{rank}\, H_i(M)$$

This is easy to prove. Indeed, $\#\mathrm{Crit}_k(f) = \mathrm{rank}\, C_k(f, X) \geq \mathrm{rank}\, HM_k$
$= \mathrm{rank}\, H_k(M)$ ∎

This is the starting point of Floer homology.

Floer homology is an infinite-dimensional version of Morse theory, and it is motivated by the following conjecture.

Conjecture (Arnold)

Let M be a cpt symplectic mfd and $H : M \times S^1 \to \mathbb{R}$ a 1-periodic Hamiltonian. Suppose that the 1-periodic solutions of $\dot{x} = X_H(x)$ are nondegenerate. Then, their number is bounded from below by

$$\sum \dim H_i(M, \mathbb{Z}_2).$$

The strategy is to construct a homology theory where (i) the chain complex is generated by 1-periodic orbits. ∂

(ii) the homology is independent of H.

(iii) it is isomorphic to Morse homology for H autonomous.

This is going to be done by a generalized Morse theory

for the action functional

$$A_H : C^\infty(S^1, M) \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto A_H(\gamma) = \int_\gamma \lambda - \int_0^1 H(\gamma(t))\,dt \,,$$

for $\omega = d\lambda$ (exact symplectic manifolds)

(For nonexact symplectic mfds, we need some technical assumptions)

The critical points of A_H are exactly the periodic orbits of X_H.

The idea is to connect critical points (orbits) by negative gradient flow lines of "grad A_H".