Symplectic Geometry I: some examples/motivations

Def. A symplectic structure on a manifold M is a 2-form $\omega \in \Omega^2(M)$ such that:

(i) ω is non-degenerate [$\omega(u, \cdot) = 0 \Leftrightarrow u = 0$)

(ii) ω is closed [$d\omega = 0$].

We call the pair (M, ω) a symplectic manifold.

Exercise: If (M,ω) is a symplectic manifold, then $\dim M = 2n$ is even.

Example: $\mathbb{R}^{2n} \ni (q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ with

$$\omega = dp \wedge dq = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \ldots + dp_n \wedge dq_n.$$

Classical Mechanics

Consider a system of Newton's equations:

$$\ddot{q} = d^2q/dt^2 = f \qquad [q \in \mathbb{R}^n]$$

[for $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $(q, dq/dt, t) \to f(q, dq/dt, t)$ a general force field].

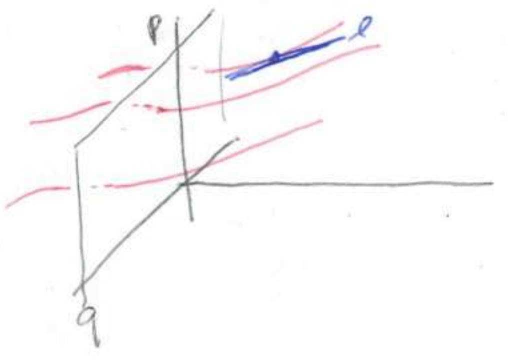
We can also write it as:

$$(*) \quad \dot{q} = p, \quad \dot{p} = f(q,p,t) \qquad [(q,p) \in \mathbb{R}^{2n}]$$

or as the line field:

$$(*) \quad \ell = \{dq = p\,dt,\ dp = f\,dt\} \quad \text{on } \mathbb{R}^{2n+1} \ni (q,p,t).$$

Where this line field ℓ prescribes the tangent lines to 'graphs' $t \mapsto (q(t), p(t), t)$ of solutions to $(*)$, as the integral curves of ℓ:



, Alternately, we can define this line field ℓ with a certain 2-form on $\mathbb{R}^{2n}\times\mathbb{R}$, namely:

$$\sigma := (dp - f\,dt)\wedge(dq - p\,dt) = \sum_{j=1}^{n} (dp_j - f_j\,dt)\wedge(dq_j - p_j\,dt)$$

Determine the line field (*) by:

$$\ell = \ker\sigma = \{ v : \sigma(v,\cdot) = (\iota_v\sigma)(\cdot) = 0 \}.$$

The closed condition (ii) $[d\omega = 0]$ for a symplectic structure is motivated by the fact that:

[The 2-form σ on $\mathbb{R}^{2n+1}$ above is closed $(d\sigma = 0)$ iff the force field f is '<u>potential</u>'.

Where we call a (possibly time dependent) force field 'potential' if there is some function $U(q,t)$, $\mathbb{R}^{n}\times\mathbb{R} \xrightarrow{U} \mathbb{R}$ s.t.

$$f = \partial_q U \quad \left[f(q,t) = (\partial_{q_1}U, \partial_{q_2}U, \dots, \partial_{q_n}U)|_{(q,t)} \right].$$

Exercise: verify σ is closed iff $f = \partial_q U$ for some $U(q,t)$.

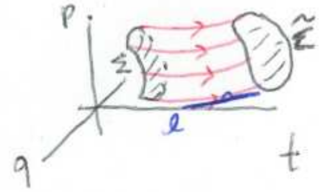
Remark: The closed 2-form σ on $\mathbb{R}^{2n+1}$ above associated to a 'potential' Newton system $\ddot{q} = \partial_q U(q,t)$ is NOT a symplectic structure [it is not non-degenerate]. It is an example of a pre-symplectic structure. Note the restrictions:

$$\sigma|_{\mathbb{R}^{2n}\times\{t_0\}} = dp\wedge dq \quad \text{are symplectic structures,}$$

which is a certain ~~typ~~ example of symplectic reduction.

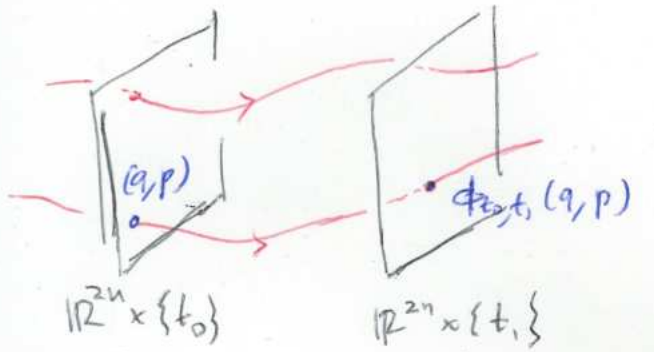
Also note that by STOKES formula, we have [when $d\sigma = 0$]

$\int_\Sigma \sigma = \int_{\tilde\Sigma} \sigma$ for any surfaces $\Sigma, \tilde\Sigma \subset \mathbb{R}^{2n+1}$ bounding a 'vortex tube' [connected by integral curves of $l = \ker\sigma$].



It follows the 'flow' of (*) is by symplectic transformations:

$$\phi^*_{t_0,t_1}(dp\wedge dq) = dp\wedge dq$$



Variation of Constants [extended Remark]

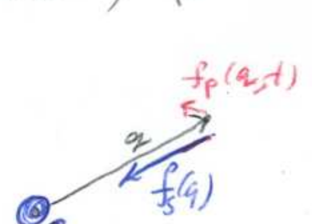
≈ (1808) Lagrange / POISSON write "1st" 'symplectic' formulas in applying method of variations of constants to study the perturbation theory of motions of the planets.

Consider motion of a given planet $q\in\mathbb{R}^3$ as solution of

$$(**)\qquad \ddot q = \underline{f_s(q)} + \underline{f_p(q,t)} = \partial_q\big(\underline{U(q)} + \underline{\Omega(q,t)}\big)$$

where f_s is the dominant force of the sun on q and f_p the (smaller) perturbing forces of the remaining planets:

Consider [given] $p(t)$



Ignoring the effects of the perturbing planets, the motion of q can be solved for exactly ('Kepler laws').

To examine the effects of these perturbing terms due to the remaining planets, Lagrange proposed to use the method of 'variation of the constants':

Consider a system of ODE's of the form:

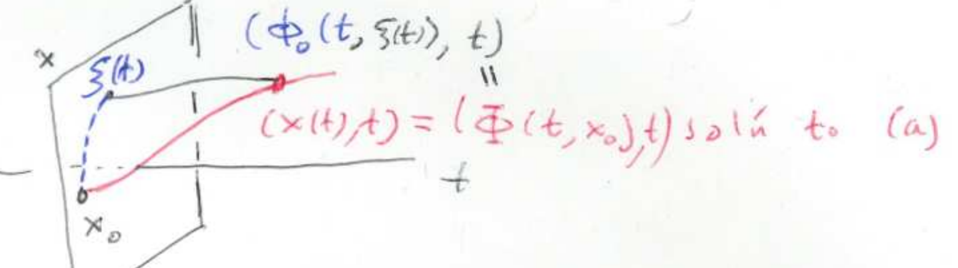
(a) $\dot{x} = v_0(x) + v_1(x,t) \quad [x \in \mathbb{R}^n]$

and suppose we <u>know</u> the general solutions $x(t) = \phi_0(t, x_0)$, $\phi_0(0, x_0) = x_0$ to $\dot{x} = v_0(x)$. Then we can seek solutions to (a) of the form:

$\tilde{x}(t) = \phi_0(t, \xi(t))$ which is equivalent to $\xi(t)$ being a sol'n of:

(b) $A(\xi,t)\,\dot{\xi} = b(\xi,t) \quad \left[A_{ij}(\xi,t)\,\dot{\xi}^j = b_i(\xi,t)\right]$

for $A(\xi,t) = \partial_x \phi_0 \big|_{(t,\xi)}$, $b(\xi,t) = v_1(\phi_0(t,\xi), t)$.



Applying this method to (∗∗), with $\xi(q,p) \in \mathbb{R}^{2n}$ certain integrals for the unperturbed (Kepler) motions, the variational eq's worked out to the following remarkable form:

$$L(\xi)\cdot\dot{\xi} = -\partial_\xi \Omega(\xi,t) \quad \left[L_{ij}(\xi)\,\dot{\xi}^j = -\frac{\partial \Omega}{\partial \xi^i}(\xi,t)\right]$$

where the matrix $L(\xi) = [L_{ij}(\xi)]$ is skew-symmetric (and invertible), ~~explicitly~~ where we can also write as:

$$\dot{\xi} = -P(\xi)\cdot\partial_\xi \Omega(\xi,t) \quad \left[\dot{\xi}^j = -P^{jk}(\xi)\cdot\frac{\partial \Omega}{\partial \xi^k}(\xi,t)\right]$$

for $P(\xi) = [P^{jk}(\xi)]$ also skew, explicitly these coefficients are:

$$L_{ij} = \sum_{\ell=1}^{n}\left(\frac{\partial p^\ell}{\partial \xi^i}\frac{\partial q^\ell}{\partial \xi^j} - \frac{\partial p^\ell}{\partial \xi^j}\frac{\partial q^\ell}{\partial \xi^i}\right) =: (\xi^i, \xi^j)$$

$$P^{jk} = \sum_{\ell=1}^{n}\left(\frac{\partial \xi^j}{\partial q^\ell}\frac{\partial \xi^k}{\partial p^\ell} - \frac{\partial \xi^k}{\partial q^\ell}\frac{\partial \xi^j}{\partial p^\ell}\right) =: \{\xi^j, \xi^k\}$$

Called the Lagrange Parentheses and Poisson Brackets respectively. The Lagrange Parentheses are exactly the coefficients of the symplectic form $dp \wedge dq$ in the coordinates $\xi(q,p) = (\xi^1, \ldots, \xi^{2n})$:

$$\omega = dp \wedge dq = \sum_{i<j} (\xi^i, \xi^j)\, d\xi^i \wedge d\xi^j .$$

[and, by definition, the Poisson brackets the coefficients of the inverse matrix].

The forms of the variational equations above follow from the following comments:

a (possibly time dependent) system:

$$\dot q = p \ , \quad \dot p = (\partial_q U)(q,t)$$

can equivalently be written as:

$$\dot q = \partial_p H \quad , \quad \dot p = -\partial_q H$$

for $H(q,p,t) = |p|^2/2 - U(q,t)$. They are a time-dependent family of symplectic gradients or Hamiltonian vector fields:

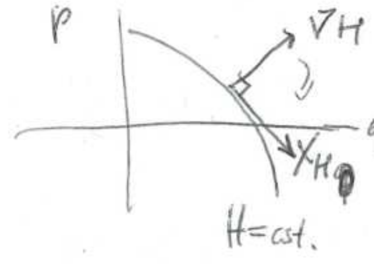
$$\iota_{X_t} \omega = - dH_t$$

for $H_t := H|_{M \times \{t\}}$ [we may also write X_{H_t} in case we need to ref H, and in case $H: M \to \mathbb{R}$ is time independent just write X_H: the symplectic gradient of $H: M \to \mathbb{R}$].

The Poisson Bracket of two functions $f, g : (M, \omega) \to \mathbb{R}$ is the function $\{f,g\} : M \to \mathbb{R}$ by:

$$\{f, g\} := \omega(X_g, X_f).$$

$\boxed{\text{Exercise}}$: for $\omega = dp \wedge dq$ on $\mathbb{R}^{2n}$ and $H(q,p)$ show that $\iota_{X_H}\omega = -dH$ determines the symplectic gradient / Hamiltonian v.f. of H as:



$$X_H = \frac{\partial H}{\partial p}\cdot \partial_q - \frac{\partial H}{\partial q}\cdot \partial_p ,$$

and the Poisson bracket of $f(q,p)$, $g(q,p)$ is:

$$\{f,g\} = \sum_{l=1}^{n}\left(\frac{\partial f}{\partial q^l}\frac{\partial g}{\partial p^l} - \frac{\partial f}{\partial p^l}\frac{\partial g}{\partial q^l}\right).$$

As well, some more properties of Poisson bracket (all of which are good exercises to show):

(i) $\{f, gh\} = \{f,g\}h + \{f,h\}g$ (Leibniz)

(ii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi ident.)

(iii) $\{f,g\} = -\{g,f\}$, $\{f+h,g\} = \{f,g\} + \{h,g\}$.

(iv) $[X_g, X_f] = X_{\{f,g\}}$.

[an operation $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to \mathbb{R}$ satisfying (i)–(iii) is called a POISSON STRUCTURE on M].

To get the variational eq's of Lagrange / Poisson, consider:

1°) $\dot f = df(X_{H_t}) = -\omega(X_f, X_{H_t}) = \{f, H_t\}$

for any $f : M \to \mathbb{R}$ (and $H(m,t)$).

2°) for $\xi^1, \dots, \xi^{2n}$ some coordinates:

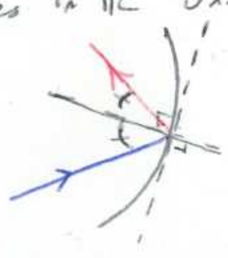
$$\dot\xi^j = \{\xi^j, H\} = -\{H, \xi^j\} = -dH(X_{\xi^j})$$

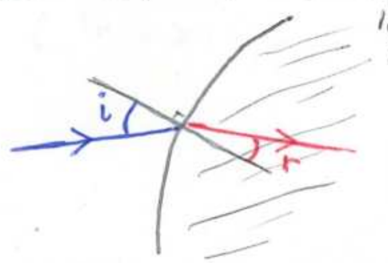
$$= -\frac{\partial H}{\partial \xi^k}\, d\xi^k(X_{\xi^j}) = -\{\xi^k, \xi^j\}\frac{\partial H}{\partial \xi^k}$$

$$= \{\xi^j, \xi^k\}\frac{\partial H}{\partial \xi^k}.$$

OPTICS

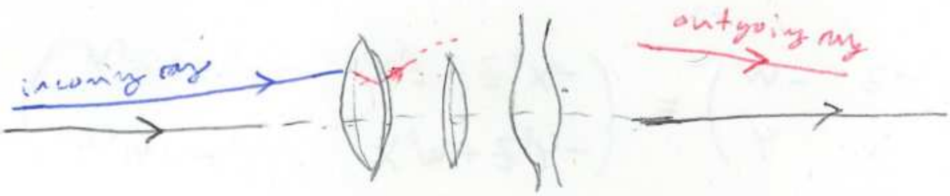
'classic' or 'geometric' optics STUDIES families of RAYS (oriented) lines in $\mathbb{R}^3$ under reflections / refractions:





$$\frac{\sin(i)}{\sin(r)} = n$$

Consider an "OPTICAL DEVICE" AS some system of refracting lenses / reflecting surfaces. Suppose, for example a rotational symmetry, so that there is a fixed axial ray (that 'passes straight through'):



We can parametrize incoming rays lying in a given plane with two variables: choose some orthogonal to the axis and assign an incoming ray to its intercept (q) / slope (p):

Then the OPTICAL SYSTEM (rotationally symmetric about the fixed axial ray) is given by some map:

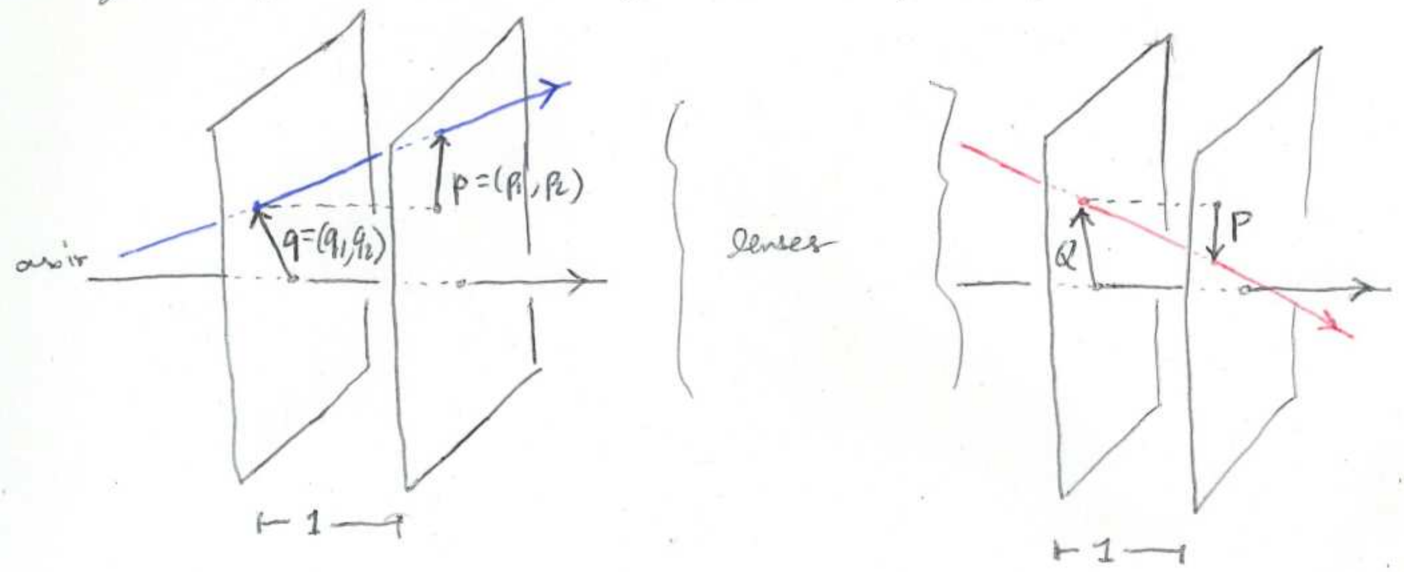
$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \quad (q,p) \mapsto (Q(q,p), P(q,p)) \qquad [\varphi(0,0) = (0,0)]$$

sending an incoming ray to the outgoing ray after refl/refr. through the system. Then, it turns out:

The linearization of φ at its fixed pt. $(0,0) = 0$ has determinant one:

$$L = d_0\varphi : \mathbb{R}^2 \circlearrowleft \quad \text{is in } SL_2(\mathbb{R}).$$

More generally (without assuming rotational symmetry):



The OPTICAL SYSTEM is described by a map:

$$\Phi : \mathbb{R}^4 \to \mathbb{R}^4 \quad (q_1, q_2, p_1, p_2) \longmapsto (Q_1, Q_2, P_1, P_2) \quad [\Phi(0)=0]$$

and its linearization at zero (the fixed axis), not only has determinant one, but is a LINEAR Symplectic map:

$$L = d_0\Phi : \mathbb{R}^4 \circlearrowleft \quad \text{is in} \quad Sp(4,\mathbb{R})$$

meaning $\omega(L\vec{u}, L\vec{v}) = \omega(\vec{u},\vec{v}) \quad \forall \vec{u},\vec{v} \in \mathbb{R}^4 \quad (\omega = dp_1\wedge dq_1 + dp_2\wedge dq_2)$.

* as we will see (next week) $\dim Sp(4,\mathbb{R}) = 10$ (whereas $\dim SL_4(\mathbb{R}) = 15$). *

An 'explanation' of this is that the space of lines in $\mathbb{R}^3$ has a symplectic structure and reflections / refractions (thru lenses) are symplectic transformations on the space of lines.

Linearizing a symplectic map (symplectomorphism) at a fixed point gives a linear symplectic map.

Some details:

lets write $\mathcal{L}$ for the space of all oriented lines in 3-dimensional Euclidean space ($\mathbb{R}^3$).

Prop. For each choice of origin $o \in \mathbb{R}^3$, there is an induced identification $\varphi_o : \mathcal{L} \longrightarrow T^* S^2$.

prf. send an oriented line ℓ, directed by unit vector u and passing through a point $p \ni \ell$ to:

$$\varphi_o(\ell) = (u, \alpha_o) \in T^* S^2$$

where $\alpha_o(\vec{v}) := (p - o) \cdot \vec{v}$ for $\vec{v} \in T_u S^2 = \{\vec{v} : u \cdot \vec{v} = 0\}$.

check: α_o is well-defined (independent of choice of $p \in \ell$), and φ_o is bi-jective. $\square$

Prop. There is a symplectic structure Ω on $\mathcal{L}$ (induced by the Euclidean structure on $\mathbb{R}^3$).

prf. There is on $T^* S^2$ (or any co-tangent bundle $T^* Q$), a canonical symplectic structure ω defined as follows. Let $\lambda \in \Omega^1(T^* S^2)$ be the 'canonical 1-form':

$$\lambda_{(u,\alpha)}(\xi) := \alpha(\pi_* \xi)$$

for $\xi \in T_{(u,\alpha)}(T^* S^2)$ and $\pi : T^* S^2 \to S^2$. Take $\omega = d\lambda$. Now, as a symplectic structure on $\mathcal{L}$, choose some origin $o \in \mathbb{R}^3$ and take $\Omega = \varphi_o^* \omega$. We claim Ω does not depend on the choice of origin. Let $o' \in \mathbb{R}^3$ be some other choice of origin with $\varphi_{o'} : \mathcal{L} \to T^* S^2$.

Then: $\alpha_{o'}(\vec{v}) = (p - o') \cdot \vec{v} = (p - o) \cdot \vec{v} + (o - o') \cdot \vec{v}$

$$= \alpha_o(\vec{v}) + d_u f(\vec{v})$$

for $f : S^2 \to \mathbb{R}$, $f(u) = (o - o') \cdot u$.

or: $\varphi_{o'} = \Psi \circ \varphi_o$, where $\Psi(u, \alpha) = (u, \alpha + d_u f)$.

In particular $\Psi : T^*S^2 \to T^*S^2$ has

$$\Psi^*\lambda = \lambda + \pi^* df = \lambda + d(\pi^* f), \text{ so that}$$

$$\varphi_{o'}^{\,*}\omega = \varphi_o^*\Psi^*\omega = \varphi_o^*\omega \quad [\Psi^*\omega = d(\Psi^*\lambda) = d\lambda = \omega]. \; \square$$

To give this symplectic structure more explicitly, we have:

$\boxed{\text{Prop:}}$ Let $U_\Sigma \subset \mathcal{L}$ be the (open set of) lines intersecting some hypersurface $\Sigma \subset \mathbb{R}^3$ transversely. Then:

① $\Omega_\ell(\delta\ell, \delta'\ell) = \delta\sigma \cdot \delta' u - \delta'\sigma \cdot \delta u$

where $\delta\ell, \delta'\ell \in T_\ell\mathcal{L}$ $(\ell \in U_\Sigma)$ are velocity vectors:

$\delta\ell = \frac{d}{dt}\big|_0 \ell(t)$ $\left[\delta'\ell = \frac{d}{dt}\big|_0 \ell'(t)\right]$ of 1-parameter families of lines directed by $u(t)$ $[u'(t)]$ in S^2 and intersecting Σ at the points $\sigma(t)$ $[\sigma'(t)]$ in Σ (and we write eg $\delta u = \dot u(0)$, $\delta\sigma = \dot\sigma(0)$)

$\boxed{\text{prf.}}$ let $\ell_0 \in U_\Sigma$ be directed by $u_0 \in S^2$ and intersect Σ at $o \in \Sigma$ (which we take as origin). Consider a 2-parameter family of lines $\ell(s,t)$ with $\ell(0,0) = \ell_0$ intersecting Σ at $\sigma(s,t)$ and directed by $u(s,t) \in S^2$. Set $\vec\sigma := \sigma - o$, so that:

$$\lambda(\partial_t \ell) = \vec\sigma \cdot \partial_t u, \quad \lambda(\partial_s \ell) = \vec\sigma \cdot \partial_s u.$$

The tangent fields $\partial_t \ell$, $\partial_s \ell$ commute so that using $\omega(X,Y) = d\lambda(X,Y) = X\cdot(\lambda(Y)) - Y\cdot(\lambda(X)) - \lambda([X,Y])$, we have:

$$\omega(\partial_s\ell, \partial_t\ell) = \partial_s\big(\lambda(\partial_t\ell)\big) - \partial_t\big(\lambda(\partial_s\ell)\big)$$

$$= \partial_s\vec\sigma \cdot \partial_t u - \partial_t\vec\sigma \cdot \partial_s u, \text{ as claimed. } \square$$

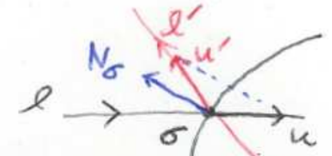
We can use ① to check that reflections (or refractions thru lenses) are symplectic transformations of $\mathcal{L}$.

Prop: Let $\Sigma \subset \mathbb{R}^3$ be a reflecting hypersurface, with $\mathcal{U}_\Sigma \subset \mathcal{X}$ the lines intersecting Σ transversely. The map

$$\mathcal{B} : \mathcal{U}_\Sigma \longrightarrow \mathcal{U}_\Sigma$$

by sending an incident ray to Σ to its reflection ('bounce' off Σ) is a symplectic transformation (symplectomorphism): $\mathcal{B}^*\Omega = \Omega$.

pf: a line $\ell \in \mathcal{U}_\Sigma$ incident to Σ at $\sigma \in \Sigma$ and directed by $u \in S^2$ is sent to $\mathcal{B}(\ell) = \ell'$ incident still at σ but now directed by $u' = u - 2(u\cdot N_\sigma)N_\sigma$ for N_σ a unit normal to Σ at σ:



by (1) above $\Omega_\ell(\delta\ell, \delta'\ell) = \underline{\delta' u\cdot\delta\sigma - \boxed{\delta u\cdot\delta'\sigma}}$ while for $\mathcal{B}_*\delta\ell$, $\mathcal{B}_*\delta'\ell$ we have the same intersection pts: $\delta\sigma = \mathcal{B}_*\delta\sigma$, but for the directions we have:

$$\delta u' = \delta u - 2\,\delta(u\cdot N)N - 2(u\cdot N)\delta N$$

where $\delta N = \frac{d}{dt}\big|_0 N_{\sigma(t)}$ $\left[\text{for } \delta\ell = \frac{d}{dt}\big|_0 \ell(t),\ \ell(t)\cap\Sigma = \sigma(t)\right]$.

Then, using $\delta\sigma\cdot N_\sigma = 0 = \delta'\sigma\cdot N_\sigma$ (since $\delta\sigma \in T\Sigma = N^\perp$),

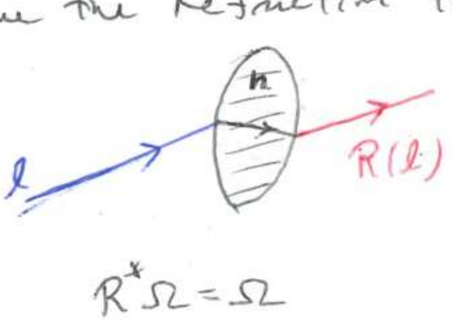
we find: $$\Omega_{\mathcal{B}(\ell)}(\mathcal{B}_*\delta\ell, \mathcal{B}_*\delta'\ell) = \Omega_\ell(\delta\ell,\delta'\ell) + 2(u\cdot N)\left[\underline{\delta N\cdot\delta'\sigma - \delta' N\cdot\delta\sigma}\right]$$

2nd fundamental form of Σ

where $\delta N\cdot\delta'\sigma = \mathrm{II}_\Sigma(\delta\sigma,\delta'\sigma) = \mathrm{II}_\Sigma(\delta'\sigma,\delta\sigma) = \delta' N\cdot\delta\sigma$

So that the underlined red terms vanish and $\mathcal{B}^*\Omega = \Omega$ as claimed. ☐

We leave the refraction through a lense case



as an exercise (it is also a symplectic map)

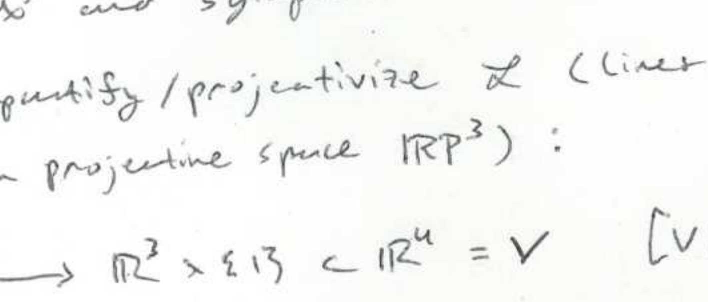
$$R^*\Omega = \Omega$$

Remark (origins of the name 'symplectic'):

A 3-parameter family of lines in $\mathbb{R}^3$ (a hypersurface in $\mathcal{L}$) is called classically a 'complex' (of lines). In particular, such a hypersurface in $\mathcal{L}$ which is given by a linear equation in 'Plücker coordinates' was called a 'linear complex' (of lines), and is closely related to linear symplectic geometry. For this reason a classic name for what we now call symplectic geometry was 'geometry of a linear complex' or 'linear complex geometry' which is now confusing (misleading at best) given we use complex geometry for geometry with complex numbers ($\mathbb{C}$). For this reason H. Weyl proposed changing the name from 'linear complex' (of lines) to 'symplectic' geometry using greek roots for an analogue of complex. It is useful though still to describe this relation between a 'linear complex' and symplectic structure:

1) Let us compactify / projectivize $\mathcal{L}$ (lines in $\mathbb{R}^3$) to $\overline{\mathcal{L}}$ (lines in projective space $\mathbb{RP}^3$):

embed $\mathbb{R}^3 \hookrightarrow \mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4 = V$ [V is a 4-dim. vectorspace].



then points of $\mathbb{R}^3$ are represented by certain lines thru the origin in V, and lines in $\mathbb{R}^3$ by certain 2-planes thru the origin of V.

we call $\mathbb{RP}^3$ = all lines thru origin of V (projective space)

$\overline{\mathcal{L}}$:= all 2-planes thru origin of V ('full' space of lines)

2) The Plücker coordinates (or embedding) on $\overline{\mathcal{L}}$ is the identification:

$$\overline{\mathcal{L}} \hookrightarrow \mathbb{P}(\mathrm{Dec.}) \subset \mathbb{P}(\Lambda^2 V) = \mathbb{RP}^5$$

by associating to each 2-plane $\Pi \in \overline{\mathcal{L}}$ $(\Pi \subset V)$

the decomposable (or 'basic') bivector

$$u \wedge v \in \Lambda^2(V) \quad \text{for} \quad \Pi = \operatorname{span}\{u, v\},$$

this bi-vector determined upto scalar multiples, so is a particular element of $\mathbb{P}(\Lambda^2(V))$ [which we denote $\mathbb{P}(\text{Dec.})$ for Dec. = decomposable bi-vectors in $\Lambda^2(V)$]

3) A 'Linear complex' (of lines) is then a 3-parameter family of lines in $\overline{\mathcal{L}}$ (a hypersurface in $\overline{\mathcal{L}}$) of the particular form:

$$\mathcal{C} = \mathcal{H} \cap \mathbb{P}(\text{Dec.})$$

where $H \subset \Lambda^2(V)$ is some hyperplane (and $\mathcal{H} = \mathbb{P}(H) \subset \mathbb{P}(\Lambda^2 V)$).

In particular any hyperplane (codimension one) in $\Lambda^2(V)$ is given by $H = \ker \omega = \{B \in \Lambda^2(V) : \omega(B) = 0\}$ for some $\omega \in \Lambda^2(V^*) \cong (\Lambda^2(V))^*$

NOTE: $\omega \in \Lambda^2(V^*)$ is exactly a skew symmetric bi-linear form on V.

As it turns out, the hypersurface $\mathcal{C}$ of lines of the 'linear complex' is a smooth hypersurface iff $\omega \in \Lambda^2(V^*)$ is non-degenerate, ie a symplectic structure on V (upto constant multiples).

4) The lines $\Pi \subset V$ of such a linear complex are what we call now the Lagrangian 2-planes of $\omega \in \Lambda^2(V^*)$:

they are the 2-planes (thru the origin of V) s.t. $\omega|_\Pi \equiv 0$.

Exercises:

1) Show a bi-vector $B \in \Lambda^2(V)$ [V a 4-dimensional vector space] is decomposable (or 'basic': $B = u \wedge v$ some $u, v \in V$) iff

$$B \wedge B = 0.$$

2) Let e_1, e_2, f_1, f_2 be a basis for V and consider the coordinates $(\alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3$ on $\Lambda^2(V)$ by:

$$\begin{aligned} B = {} & (\alpha_1 + \beta_1)\, e_1 \wedge e_2 + (\alpha_1 - \beta_1)\, f_1 \wedge f_2 \\ & + (\alpha_2 + \beta_2)\, e_2 \wedge f_1 + (\alpha_2 - \beta_2)\, e_1 \wedge f_2 \\ & + (\alpha_3 + \beta_3)\, e_1 \wedge f_1 + (\alpha_3 - \beta_3)\, f_2 \wedge e_2 . \end{aligned}$$

Check that the decomposable bi-vectors are given by:

$$(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = (\beta_1)^2 + (\beta_2)^2 + (\beta_3)^2 .$$

3) Let e^1, e^2, f^1, f^2 be the dual basis of e_1, e_2, f_1, f_2 and consider the symplectic form $\omega = f^1 \wedge e^1 + f^2 \wedge e^2$. Call a decomposable bi-vector $B \in \Lambda^2(V)$ Lagrangian if $\omega(B) = 0$. Check that in the coordinates of (2) ~~that~~ B is Lagrangian iff $\beta_3 = 0$.

4) Let $Gr(2, V)$ [$Gr^+(2, V)$] be the 2-planes [oriented 2-planes] through the origin of V, and $\Lambda(2)$ [$\Lambda^+(2)$] the Lagrangian 2-planes [oriented Lagrangian 2-planes] through the origin of (V, ω). Deduce from (2), (3) that:

$$Gr^+(2, V) \approx S^2 \times S^2 , \quad Gr(2, V) \approx S^2 \times S^2 / _{(x,y) \sim (-x,-y)}$$

$$\Lambda^+(2) \approx S^2 \times S^1 , \quad \Lambda(2) \approx S^2 \times S^1 / _{(x,y) \sim (-x,-y)} .$$

* in general we call n-dimensional subspaces through the origin of $(\mathbb{R}^{2n}, dp \wedge dq)$ on which ω restricts to zero, Lagrangian subspaces, and $\Lambda(n)$ the Lagrangian Grassmannian.

(15)

We can ask: how do the lines of a 'linear complex' look like in $\mathbb{R}^3$? Answer (figure it out as an exercise):

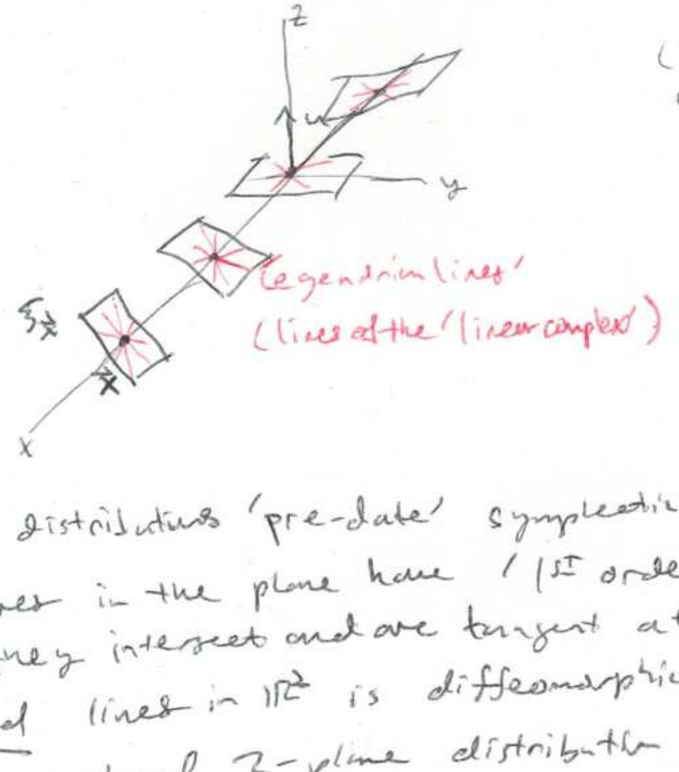
let $u \in S^2$ be a unit vector, and at each $\vec{x} \in \mathbb{R}^3$ consider the plane $\xi_{\vec{x}} \ni \vec{x}$ through $\vec{x}$ with normal:

$$N_{\vec{x}} = \vec{x} \times u + u \qquad \left[\text{ie } \xi_{\vec{x}} = \vec{x} + (N_{\vec{x}})^{\perp} \right]$$

the 3-parameter family of lines of the linear complex are the lines: $\ell = \vec{x} + \langle v \rangle$ for $v \in \xi_{\vec{x}}$.

We call the distribution $\xi_{\vec{x}}$ of planes on $\mathbb{R}^3$ a contact distribution (and the lines of the 'linear complex' are the 'Legendrian lines' of this contact distribution:

in general given a plane distribution ξ we can call its 'Legendrian curves' those curves which are tangent to ξ:



(the picture is symmetric wrt 'vertical' translations along the z-axis and rotation about the z-axis).

* Contact distributions 'pre-date' symplectic structures: two curves in the plane have '1st order contact' at a given point if they intersect and are tangent at said point. The space of pointed lines in $\mathbb{R}^2$ is diffeomorphic to $\mathbb{R}^2 \times S^1 = \mathbb{P}(T\mathbb{R}^2)$, and has a natural 2-plane distribution (contact structure) as the span of tangents to curves. In coordinates, we take $(x,y) \in \mathbb{R}^2$ and $p \in \mathbb{R}$ as the slope of a line through (x,y). then our contact distribution is given by:

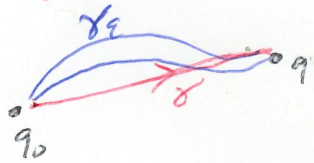
$$\xi_{(x,y,p)} = \{ dy = p\, dx \}. \quad *$$

Variational Principles

In OPTICS (or Riemannian manifolds (M, g)) we consider light rays (unparametrized geodesics of $\mathbb{R}^3, dx^2+dy^2+dz^2$). They have variational characterization as extremals of:

- length functional: $\gamma \mapsto \int |\dot\gamma| dt = \text{length}(\gamma)$

among eg fixed endpoint curves $\gamma : q_0 \to q_1$ (q_0, q_1 fixed)

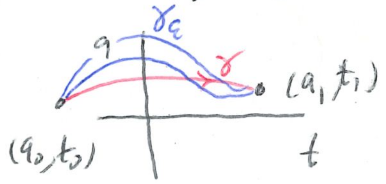


[extremals are unparametrized geodesics].

In classical mechanics we also have a variational principle: trajectories of $\ddot q = \partial_q U(q,t)$ as extremals of

- Action functional: $\gamma \mapsto \int L(\gamma, \dot\gamma, t)\, dt = A(\gamma)$

among eg fixed times $t_0 < t_1$ and endpoints q_0, q_1 variations,



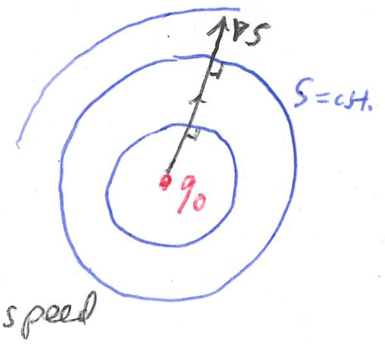
where $L(q, v, t) = \frac{|v|^2}{2} + U(q,t)$ $(v = \dot q)$

is the Lagrangian function.

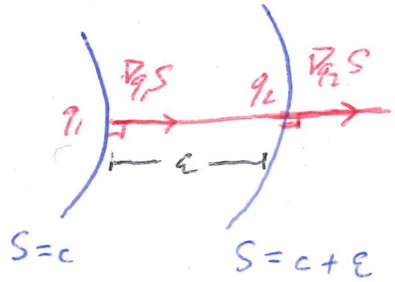
In Riemannian geometry (eg $\mathbb{R}^3, dx^2+dy^2+dz^2$) we have:

1) fix q_0 and set $S(q) := \text{dist}(q_0, q) = \inf\limits_{\gamma: q_0 \to q} \text{length}(\gamma)$,

whose level sets are the spheres centered at q_0.



2) The radial lines from q_0 follow the normals to these spheres, in fact, integral curves of: $\dot q = \nabla_q S$ are the radial unit speed geodesics 'emitted' from q_0;



$q_2 = q_1 + \varepsilon \nabla_{q_1} S$

ie

$\{S = c+\varepsilon\} = \{q + \varepsilon \nabla_q S : S(q) = c\}$.

$\varepsilon = \text{dist}(S_c, S_{c+\varepsilon})$.

Conversely, if $S : \mathbb{R}^3 \to \mathbb{R}$ is a solution of the

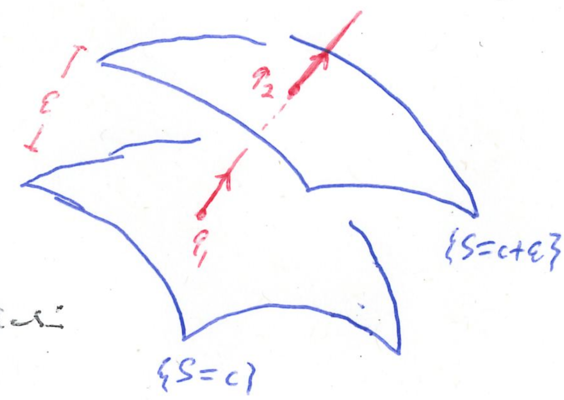
$\boxed{\text{EIKONAL eq}^{\underline{n}}}$ $\quad |\nabla S|^2 = 1$

then (regular, cpt) level sets have (at least for ε not too large)

$$\text{dist}(\{S=c\}, \{S=c+\varepsilon\}) = \varepsilon$$

and the solutions of the 1^{st} order ODE:

$$(*) \qquad \dot{q} = \nabla_q S$$



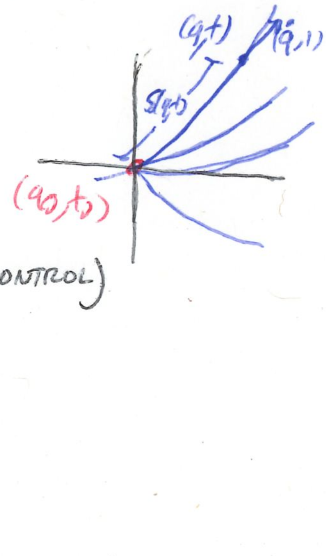
by 'following the normals' are unit speed geodesics

$\{S = c+\varepsilon\} = \{q + \varepsilon \nabla_q S : S(q) = c\}$ still.

$\boxed{\text{Remark}}$: Analytically the 2^{nd} order ODES (geodesic eq$^{\underline{s}}$) can be 'split' into a 1^{st} order PDE (EIKONAL) with a corresponding 1^{st} order ODE $(*)$.

In classical mechanics, replacing distance by Action, we find a similar situation (Hamilton's 'OPTICAL ANALOGY'):

1) fix some q_0, t_0, and set $S(q,t) = \inf\limits_{\gamma : (t_0, q_0) \to (t_1, q_1)} A(\gamma)$



2) Then (see e.g. M. Levi: Calculus of variations & OPTIMAL CONTROL) $S(q,t)$ satisfies:

$$(*) \quad \left[\begin{array}{l} \partial_q S \cdot \dot{q} + \partial_t S = L \\ \partial_q S(q,t) = \partial_v L(q,\dot{q},t) \end{array}\right.$$

the equation $\partial_q S = \partial_v L$ is a 1^{st} order ODE satisfied by trajectories.

3) In case, for each given (q,t), the equation

$$p = \partial_v L(q, \dot{q}, t) \qquad [\text{Legendre transform}]$$

is invertible to determine $\dot{q}(q,p,t)$, we can then set $H(q,p,t) := p\cdot\dot q - L(q,\dot q,t)$

[substituting $\dot q(q,p,t)$], as the corresponding Hamiltonian. Then the eq's (*) have the form:

$\boxed{\text{HAMILTON-JACOBI}}$ $$H(q,\partial_q S,t) + \partial_t S = 0 \qquad (H\text{-}J)$$

and solutions of the 1st order ODE

$$(*) \qquad p = \partial_v L(q,\dot q,t) = \partial_q S(q,t)$$

are trajectories of the Newton system.

4) Conversely if $S(q,t)$ satisfies the 1st order PDE (H-J) the solutions of 1st order ODE (*) are trajectories, with

$$\mathcal{A}(q_*)_{t_0,t_1} = S(q_*(t_1),t_1) - S(q_*(t_0),t_0)$$

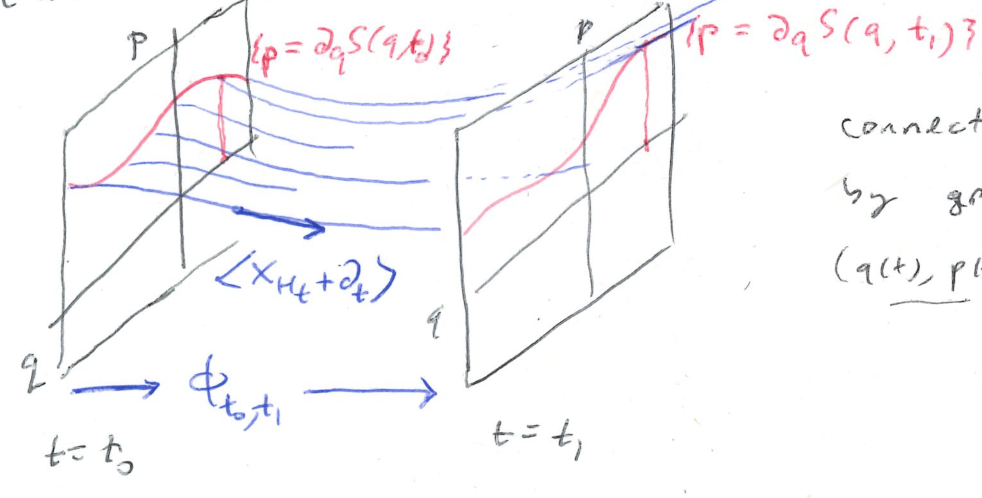
along such a trajectory q_* solving (*).

$\boxed{\text{Exercise}}$: Supposing that for each (q,t), $p = \partial_v L(q,\dot q,t)$ is invertible to determine $\dot q(q,p,t)$, check that the Euler-Lagrange eq's

$$\frac{d}{dt}\big(\partial_v L(q,\dot q,t)\big) = \partial_q L(q,\dot q,t) \qquad (E\text{-}L)$$

for extremals of L are, for $H(q,p,t) = p\cdot\dot q - L(q,\dot q,t)$ given by Hamilton's eq's: $$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q}.$$

To describe this situation in 'symplectic' language, consider that on $\mathbb{R}^{2n}\times\mathbb{R} \ni (q,p,t)$ we ~~have~~ an evolving family of graphs:



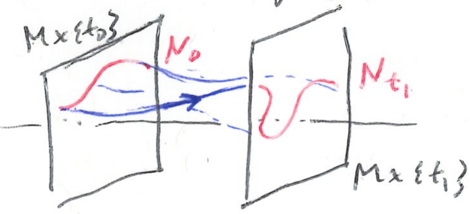
connected (ODE (*)) by graphs of solutions $(q(t),p(t),t)$ tan. to $\langle X_{H_t} + \partial_t\rangle$

* Note that, at each fixed $t = t_0$, the submanifold $N_{t_0} = \{(q, \partial_q S(q,t_0)) : q \in \mathbb{R}^n\}$ is a Lagrangian submanifold of $\mathbb{R}^{2n}$, $dp \wedge dq$:

$$\lambda\big|_{TN_{t_0}} = p \cdot dq\big|_{TN_{t_0}} = d(S|_{t=t_0}) \rightarrow \omega\big|_{TN_{t_0}} = d\lambda\big|_{TN_{t_0}} = 0. \; *$$

$\boxed{\text{Exercise:}}$ Consider T^*Q, with its standard symplectic structure $\omega = d\lambda$ $[\lambda_{(q,p)}(\xi) = p(\pi_*\xi)]$. Let $\alpha : Q \to T^*Q$ a 1-form on Q, with 'graph' $\Gamma_\alpha = \mathrm{im}(\alpha) \subset T^*Q$. Check that $\Gamma_\alpha \subset T^*Q$ is Lagrangian iff $d\alpha = 0$ is closed, [in particular if $S : Q \to \mathbb{R}$ is a function then $\Gamma_{dS} \subset T^*Q$ is Lagrangian].

So, the H-J equation is describing a particular case of the following situation: let $\phi_{t_0,t} : M \circlearrowleft$ $(\phi_{t_0,t}{}^*\omega = \omega)$ be the flow of $X_{H_t} \in \mathfrak{X}(M)$ the Ham. vf's of some $H : M \times \mathbb{R} \to \mathbb{R}$. Suppose $N_0 \subset M$ is a Lagrangian submanifold, and set $N_t = \phi_{t_0,t}(N_0) \subset M$, as well a Lagrangian submanifold. then $\hat{N} = (N_t, t) \subset M \times \mathbb{R}$ corresponds to the 'graphs' foliated by trajectories of a sol'n to (H-J)



In the particular case that (M,ω) is an exact symplectic manifold: $\omega = d\lambda$ some one-form λ on M, then we may take:

① $\alpha = \lambda - H\,dt$ as the 'action' 1-form on $M \times \mathbb{R}$

[Note: $\ker d\alpha = \langle X_{H_t} + \partial_t \rangle$ and extremals (e.g. fixed endpts (x_0,t_0), (x_1,t_1) $\in M\times\mathbb{R}$) of $\gamma \mapsto \int_\gamma \alpha$ are graphs of trajectories: $(x(t),t)$, $\dot{x} = X_{H_t}$.

(2) an $n+1$ dimensional submanifold (connected say)

$\hat{N} \subset M \times \mathbb{R}$ has $d\alpha|_{T\hat{N}} \overset{(*)}{\equiv} 0$

iff $N_t = \hat{N} \cap (M \times \{t\}) \subset M$ is a Lagrangian submfld, and

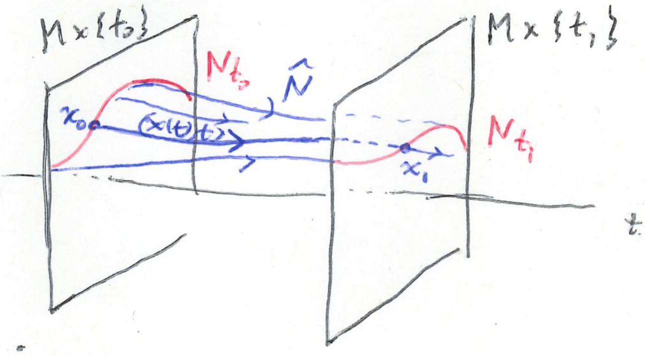
$\phi_{t_0,t_1}(N_{t_0}) = N_{t_1}$ $\quad (\langle X_{H_t} + \partial_t \rangle \subset T\hat{N})$.

So, the eqn. (*) characterizes 'graphs' in $M \times \mathbb{R}$ of Lagrangian submflds in M evolving under the flow $\phi_{t_0,t_1} : M \circlearrowleft$ of X_{H_t}.

(3) In particular, if we happen to have

$\alpha|_{\hat{N}} = dS$ for some function $S : \hat{N} \to \mathbb{R}$

then this S plays the role of a solution to (H-J), as well as an 'action function':



$S(x_1,t_1) - S(x_0,t_0)$

$= A(x)_{t_1,t_0} = \int_{\{(x(t),t)\,:\,t_0 \le t \le t_1\}} \overbrace{\lambda - H\,dt}^{\alpha}$.