Ex. 1

S_{ε^*} is of c_0-Lipschitz type if

$\exists\, p, L > 0$ s.t

$$c(\varepsilon) \le c(\varepsilon^*) + L(\varepsilon - \varepsilon^*)$$

$$\overset{\shortparallel}{c_0(B_\varepsilon, \omega)} \qquad \forall\, \varepsilon^* \le \varepsilon \le \varepsilon^* + p$$

where B_ε is s.t $\partial B_\varepsilon = S_\varepsilon$

(and $\varepsilon \le \varepsilon' \Rightarrow B_\varepsilon \subseteq B_{\varepsilon'}$)

1. Let $(\tilde{S}_v)$ be another parametrized family (and $\tilde{B}_v$ s.t $\partial \tilde{B}_v = \tilde{S}_v$), s.t $\tilde{S}_{\varepsilon^*} = S_{\varepsilon^*}$

wlog, take $\varepsilon^* = 0$

Claim: $\exists\, 0 < c < 1$ s.t:

$$\forall v \in [0, p]: \ \tilde{B}_{cv} \subseteq B_v$$

Pf: exercise (choose a nbhd of $S, \tilde{S}$ and use cptness)

$\therefore$

c_0-Lipschitzness of S_0

$$\tilde{c}(v) \leq c\left(\frac{v}{c}\right) \leq c(0) + L\left(\frac{v}{c}\right)$$

monotonicity

$$= \tilde{c}(0) + \underbrace{\frac{L}{c}}_{=:\tilde{L}} \cdot v \qquad \square$$

2) A nhbd of S is parametrized by the flow φ^t of the Liouville v.f V (see lectures. Indeed, by integrating $\mathcal{L}_V \omega = \omega$, we get $(\varphi^t)^* \omega = e^t \omega$)

$$\Rightarrow \quad c_0(B_\varepsilon, \omega) = c_0(B_0, e^\varepsilon \omega) = e^\varepsilon c_0(B_0, \omega)$$

$\therefore \quad c(\varepsilon) = e^\varepsilon c(0)$ / Therefore, we have c_0-Lipschitz continuity near $\varepsilon = 0$. $\square$

Ex 2.

1. ∂W is of restricted contact-type iff the Liouville vector field is (positively) transverse to ∂W.

2. The Liouville v.f is s.t $V \lrcorner \omega = \lambda_0$.

For $\mathbb{D}^{2n} \subset \mathbb{R}^{2n}$, we have

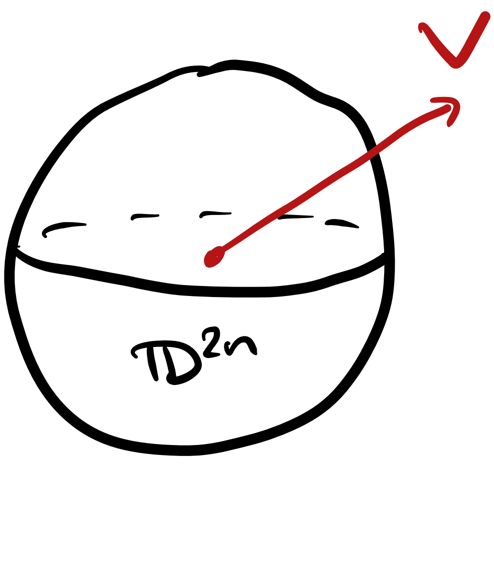
$$\omega_0 = \sum dq_i \wedge dp_i$$

$$\lambda_0 = \frac{1}{2} \sum q_i \, dp_i - p_i \, dq_i$$

Write $V = \sum_i a_i \partial_{q_i} + b_i \partial_{p_i}$

$$V \lrcorner \omega = \sum_i a_i dp_i - b_i dq_i$$

$\Rightarrow$ Liouville v.f : $V = \sum_i q_i \partial_{q_i} + p_i \partial_{p_i}$



This is clearly positively transverse to $\partial \mathbb{D}^{2n} = \mathbb{S}^{2n-1}$ (it's in the radial direction, so it's even normal to it).

3. <u>W.T.S</u> $\partial(\mathbb{D}^*N) = \mathbb{S}^*N$ as manifolds.

Let $\{\mathcal{U}_i, \psi_i\}$ be an atlas for N.

Refining it if necessary, we have trivializations

$$\begin{tikzcd} \mathbb{D}^*\mathcal{U}_i \arrow[r, "\cong"] \arrow[d] & \mathcal{U}_i \times \mathbb{D}^n \arrow[dl] \\ \mathcal{U}_i & \end{tikzcd}$$

But now, it is std knowledge that $\mathbb{D}^n$ is a mfd with boundary $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$!

So we can choose an atlas $\{V_j, \phi_j\}$ for $\mathbb{D}^n$ (which consists of both interior and boundary charts). So this gives us an atlas

$$\{(\mathcal{U}_i \times V_j, \psi_i \oplus \phi_j)\}$$

for $\mathbb{D}^*N$ (where $\psi_i \oplus \phi_j$ is an interior/bdry chart depending on whether ϕ_j is).

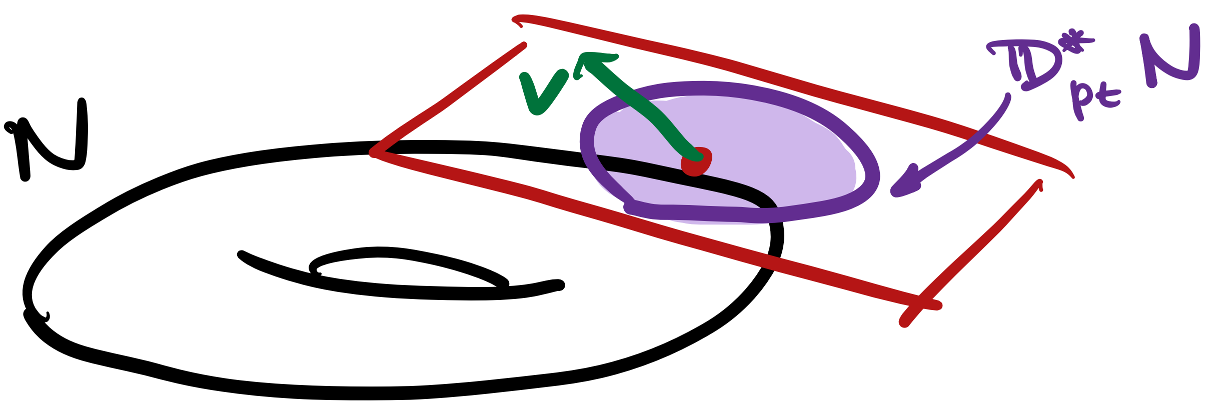
4. Std. symplectic form on T^*N:

$$\omega = \sum_i q_i \wedge p_i$$

$$\lambda = -\sum p_i \, dq_i$$

Liouville v.f.: $\underbrace{V = \sum p_i \partial_{p_i}}$

This is positively transverse to $\underbrace{\|p\|=1}_{\partial \mathbb{D}^n}$



Ex. 3

1.

- $W = \mathbb{D}^{2n}$

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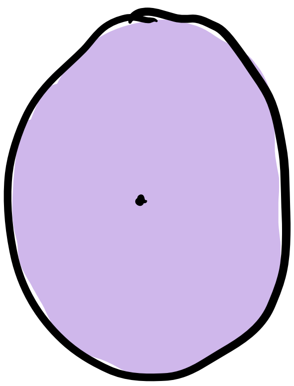
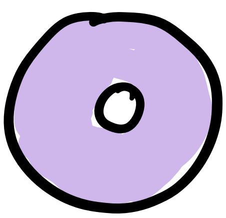
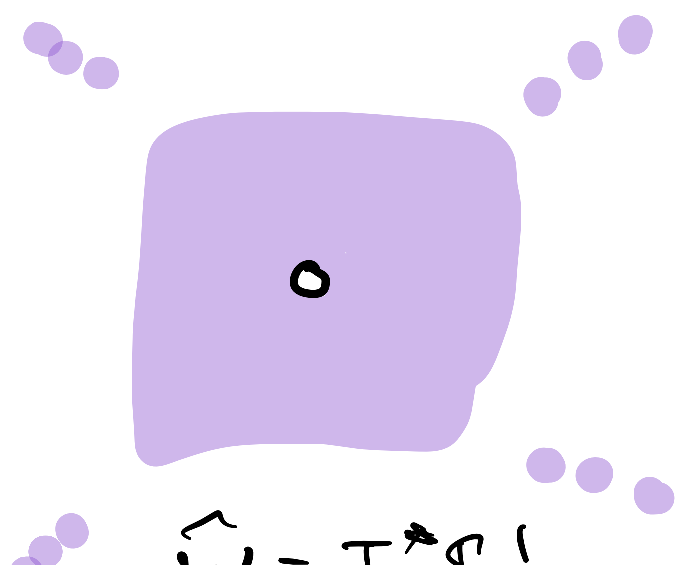


- $W = \mathbb{D}^* S^1$

a) in $\mathbb{R}^2$:

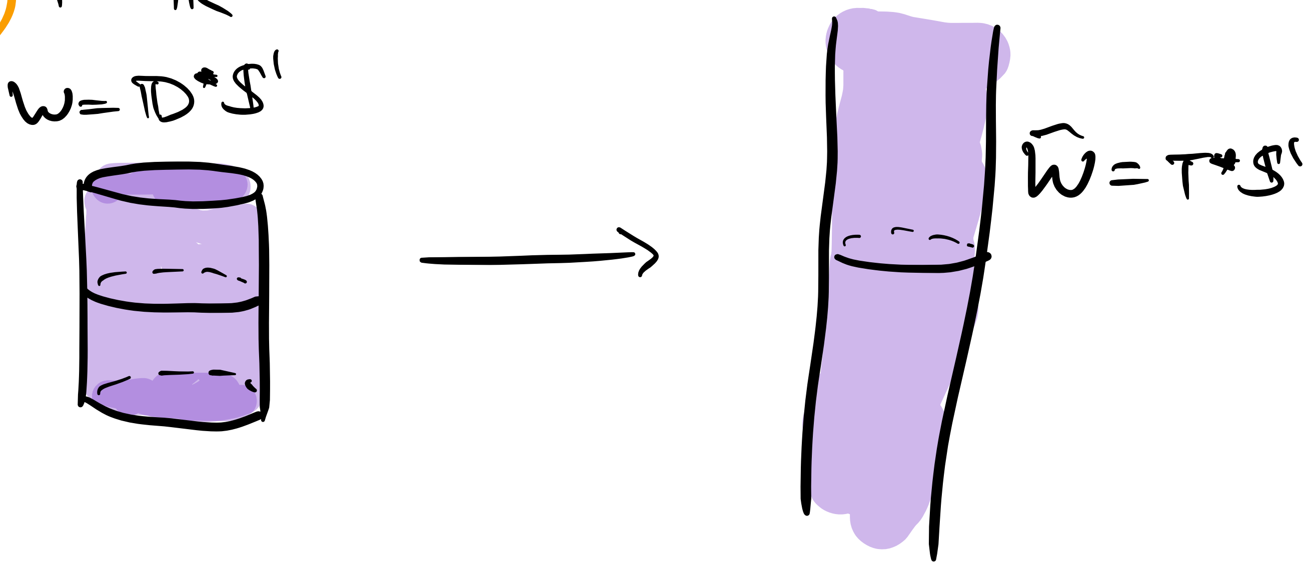


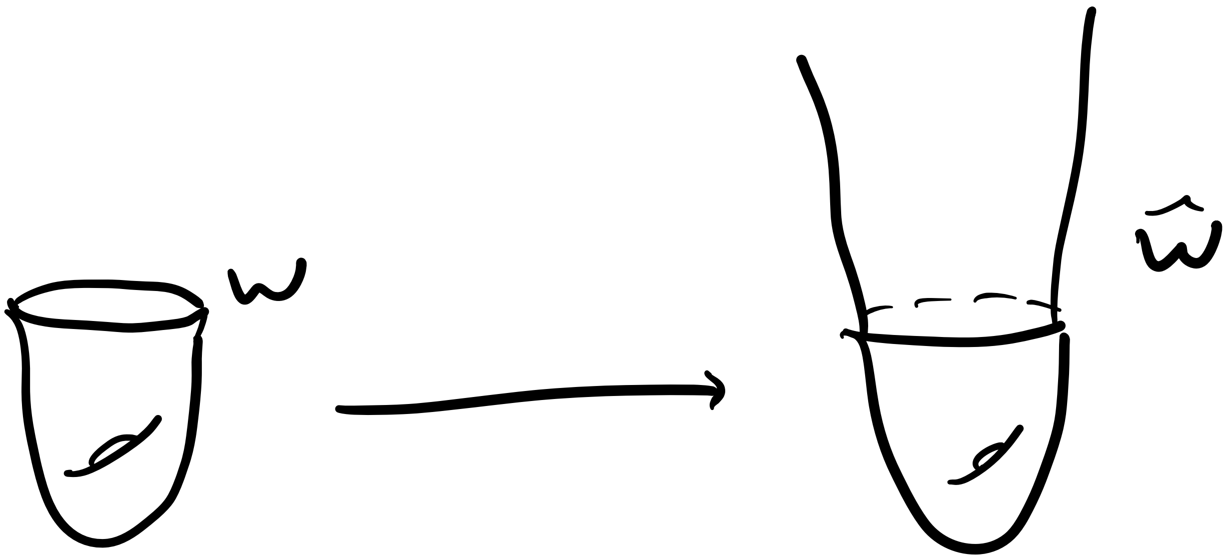
$W = \mathbb{D}^* S^1$



$\hat{W} = T^* S^1$

b) in $\mathbb{R}^3$:





2. Let $H : \hat{W} \to \mathbb{R}$ be linear at infinity i.e.

$$\forall r \geq r_0 : \quad H = H(r) = ar + b$$

$$\Longrightarrow X_H = J\nabla H$$
$$= J(a\partial_r)$$
$$= aR$$

$J\partial_r = R$

3. For $r \geq r_0$, X_H is proportional to the Reeb vector field $\Rightarrow$ trajectories of X_H are simply reparametrizations of Reeb trajectories on ∂W.
In particular, they are constrained to slices $\{r\} \times \partial W$.

4. Since $X_H = aR$, then:

$$\left\{ \begin{array}{c} \text{Ham. orbits} \\ x(t) \ \ (\dot{x}(t) = X_H) \\ \text{on } \{r\} \times \partial W \\ (r \geq r_0) \end{array} \right\} \xleftrightarrow{\text{Bijection}} \left\{ \begin{array}{c} \text{Reeb orbits} \\ y(t) \ \ (\dot{y}(t) = R) \\ \text{on } \partial W \end{array} \right\}$$

$$x(t) \longmapsto y(t) := x\left(\frac{t}{a}\right)$$

Period 1 Ham. orbit.

Period a Reeb chord

If $a \notin \mathrm{Spec}\,\alpha$, then there can be no Ham. orbits above $r \geq r_0$.

Ex 4.

Idea of Moser's trick:

assume existence of an isotopy btwn

$$\underbrace{\psi^* f}_{\substack{\shortparallel \\ g_0}} \quad \text{and} \quad g_1 = \underbrace{x_1^2 + \dots + x_j^2 - x_{j+1} - \dots - x_n^2}_{(\text{assume } f(p)=0)}$$

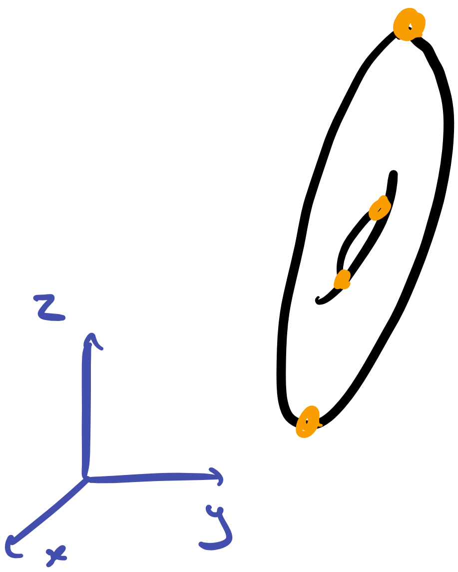
Set $g_t := (1-t)g_0 + t g_1$

Then, want $\underbrace{\frac{d}{dt} g_t}_{\substack{\shortparallel \\ g_1 - g_0 + \mathcal{L}_{X_t} g_t}} = 0$

Write ϕ_t = flow of X_t. Then, need:

$\mathcal{L}_{X_t} g_t = g_0 - g_1 = \psi^* f - x_1^2 - \dots - x_j^2 + x_{j+1}^2 + \dots + x_n^2$ solution exists by classical theorems $\square$

Ex. 5



$M = T^2 \hookrightarrow \mathbb{R}^3$

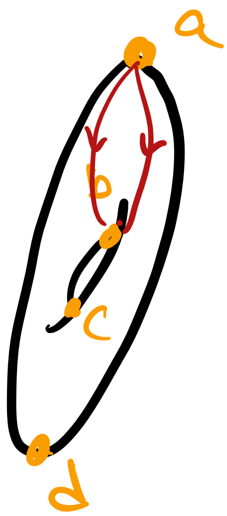
$$\left.\begin{array}{l} f : \mathbb{R}^3 \longrightarrow \mathbb{R} \\ (x,y,z) \longmapsto z \end{array}\right\}$$ height function, which we restrict to the torus and still call f

1. → Crit. points in orange.

(Indices: 2 for the max. 0 for the min. 1 for the saddle points)

• **Note:** The reason we tilt the torus, instd of keeping it vertical, is to ensure there are no flow lines connecting crit. points of same indices (the saddle points, here).

Informally, this is the Morse-Smale condition.

2. 

Let's give names to the crit points.

f = height function

So "flow lines of $-\nabla f$"

= "trajectories of fastest descent".

There are 2 from a to b

2 from a to c

0 from b to c

2 from b to d

2 from c to d

(and a 1-dim. family of trajectories from a to d, but we don't care abt those here).

So, chain complex (over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$):

$$CM_0 = \mathbb{Z}_2\langle d \rangle \simeq \mathbb{Z}_2$$

$$CM_1 = \mathbb{Z}_2\langle b, c \rangle \simeq \mathbb{Z}_2^2$$

$$CM_2 = \mathbb{Z}_2\langle a \rangle = \mathbb{Z}_2$$

- $\partial : CM_2 \to CM_1$ defined by

$$\partial a = 2b + 2c = 0 \quad (\text{in } \mathbb{Z}_2)$$

- $\partial : CM_1 \to CM_0$ defined by

$$\begin{cases} \partial b = 2d = 0 \\ \partial c = 2d = 0 \end{cases}$$

So: • $HM_0 = \dfrac{\ker\{\partial : CM_0 \to CM_{-1}\}}{\operatorname{im}\{\partial : CM_1 \to CM_0\}} = \dfrac{\mathbb{Z}_2}{0} = \mathbb{Z}_2$

• $HM_1 = \dfrac{\ker\{\partial : CM_1 \to CM_0\}}{\operatorname{im}\{\partial : CM_2 \to CM_1\}} = \dfrac{\mathbb{Z}_2^2}{0} = \mathbb{Z}_2^2$

• $HM_2 = \dfrac{\ker\{\partial : CM_2 \to CM_1\}}{\operatorname{im}\{\partial : CM_3 \to CM_2\}} = \mathbb{Z}_2$

This indeed agrees with the singular homology of the torus, which counts the number of holes. (see Hatcher, or most any book on algebraic topology)

$\square$