Problem Sheet 8 Solutions.

Ex 1 1. Consider a basis $\{e_i, f_i\}$ s.t

$$\begin{cases} \langle e_i, e_i \rangle = \langle f_i, f_i \rangle \\ \text{Basis is orthogonal.} \end{cases}$$

Let $A = \begin{pmatrix} \uparrow & & \uparrow & \uparrow & & \uparrow \\ e_1 & \cdots & e_n & f_1 & \cdots & f_n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{pmatrix} \in Sp(2n)$

true since such a basis is symplectic w.r.t $g_i = \omega(\cdot, \cdot)$

Then: $g(A(x,y), A(x,y)) = g\left(\sum_j x_j e_j + y_j f_j, \sum_j x_j e_j + y_j f_j\right)$

$$= \sum_j \langle e_j, e_j \rangle (x_j^2 + y_j^2)$$

>0, so can write it $\frac{1}{r_j^2}$

Wlog, $r_1 \le r_2 \le \dots \le r_n$ (else, re-order).

$\therefore A \cdot E(g) \longrightarrow E(r)$

$\{g < 1\} = A\cdot E(g)$, $E(r) = \{\dots \sum_j \frac{1}{r_j^2}(x_j^2 + y_j^2) < 1\}$

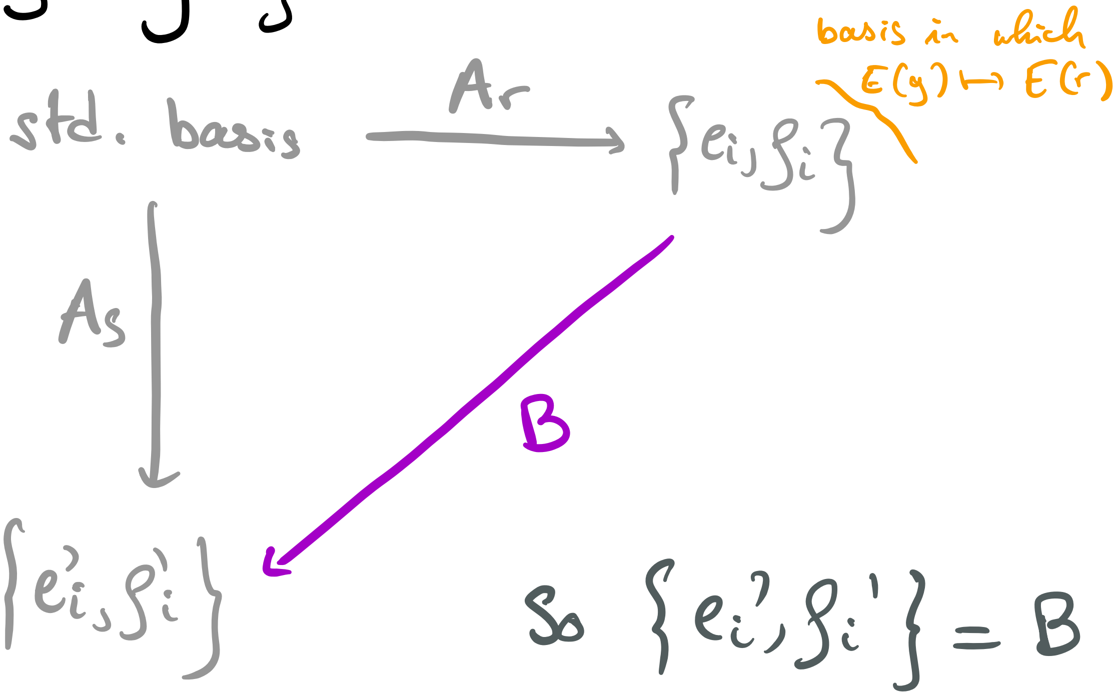
$\square$

2. By 1., $\exists \begin{cases} A_r \\ A_s \end{cases} \in Sp(2n)$ s.t.

$$\left.\begin{aligned} A_r E(g) &= E(r) \\ A_s E(g) &= E(s) \end{aligned}\right\} \Rightarrow A_r^{-1} E(r) = A_s^{-1} E(s) \Rightarrow E(s) = B\,E(r)$$

$$\left(B := A_s A_r^{-1} \in Sp(2n)\right)$$

By def^n, here is what our matrices do:



So $\{e_i', f_i'\} = B\,\{e_i, f_i\}$

$\rightsquigarrow$ in the basis $\{e_i, f_i\}$, g is given by

$$\underbrace{\begin{pmatrix} \langle e_i, e_i \rangle & & \\ & \ddots & \\ & & \langle f_i, f_i \rangle \end{pmatrix}}_{D_r} = \begin{pmatrix} \frac{1}{r_1^2} & & \\ & \ddots & \\ & & \frac{1}{r_n^2} \end{pmatrix}$$

$\rightsquigarrow$ in the basis $\{e_i', f_i'\}$, must have:

$$D_s = C D_r C^{-1} = \begin{pmatrix} \langle e_i', e_i' \rangle & & \\ & \ddots & \\ & & \langle f_i', f_i' \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s_1^2} & & \\ & \ddots & \\ & & \frac{1}{s_n^2} \end{pmatrix}$$

$\Rightarrow$ both matrices are similar

$\Rightarrow$ same eigenvalues

$\Rightarrow$ $\left\{ \frac{1}{r_i^2} \right\} = \left\{ \frac{1}{s_i^2} \right\}$ $\square$

Ex 2 (Isoparametric inequality)

$(V, \omega) \cong (\mathbb{C}^n, \omega_0)$ (wlog)

$$A(\gamma) = \frac{1}{2}\int_0^1 \omega(\dot\gamma(t), \gamma(t))\, dt$$

$$E(\gamma) = \frac{1}{2}\int_0^1 \|\dot\gamma(t)\|^2 dt$$

$$L(\gamma) = \int_0^1 \|\dot\gamma(t)\|\, dt$$

(definitions)

W.T.S $\left(|A(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2 \leq \frac{1}{2\pi} E(\gamma) \right)$

1) Let's show that $|A(\gamma)| \leq \frac{1}{2\pi} E(\gamma)$

Write down: $\underbrace{\gamma(t) = \sum_{k\in\mathbb{Z}} e^{2\pi J_0 k t} a_k}_{\text{Fourier decomposition}}$ $(a_k \in \mathbb{C}^n)$

Then, $\dot\gamma(t) = \sum_{k\in\mathbb{Z}} 2\pi k J_0 e^{2\pi J_0 k t} a_k$

$$2A(\gamma) = \int_0^1 \omega(\dot{\gamma}(t), \gamma(t))\, dt$$

$\omega(\cdot, J_0 \cdot) = g(\cdot, \cdot)$

$$= \int_0^1 \langle \dot{\gamma}(t), J_0 \gamma(t) \rangle \, dt$$

$$= \int_0^1 \sum_k 2\pi k \langle J_0 e^{2\pi J_0 k t} a_k , J_0 \gamma(t)) \, dt$$

J an isometry

$$= \int_0^1 \sum_{k,l} 2\pi k \langle e^{2\pi J_0 k t} a_k, e^{2\pi J_0 l t} a_l \rangle \, dt$$

$$= \sum_{k,l} 2\pi k \underbrace{\int_0^1 \langle e^{2\pi J_0 k t} a_k , e^{2\pi J_0 l t} a_l \rangle \, dt}_{= \delta_{kl} \|a_k\|^2}$$

(std. Fourier analysis)

$$\Longrightarrow A(\gamma) = \pi \sum_k k \|a_k\|^2$$

✓

Now compute:

$$E(\gamma) = \frac{1}{2}\int_0^1 \|\dot{\gamma}\|^2\,dt$$

$$= \frac{1}{2}\int_0^1 \sum_{k,l} 4\pi^2 kl \langle e^{2\pi J_0 kt} a_k, e^{2\pi J_0 lt} a_l \rangle\,dt$$

$$= 2\pi^2 \sum k^2 \|a_k\|^2$$

- $$A(\gamma) = \pi \sum_k k\|a_k\|^2$$

$\therefore$

$$\frac{1}{2\pi} E(\gamma) = \pi \sum_k k^2 \|a_k\|^2$$

And $|A(\gamma)| = \left|\pi \sum k\|a_k\|^2\right| \leq \pi \sum_k |k|\cdot\|a_k\|^2$

$|k| \leq k^2$
$\forall k \in \mathbb{Z}$

$$\leq \pi \sum_k k^2 \cdot \|a_k\|^2$$

$$= \frac{1}{2\pi} E(\gamma)$$

$\square$

2) Let's show $|A(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2$

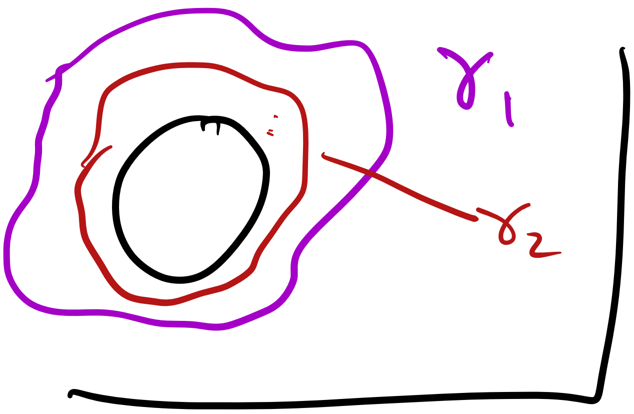
$$\gamma : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}^n$$

Wlog, $L(\gamma) = 1$ (else, replace γ by $\frac{\gamma}{L(\gamma)}$)

possible since we're in a vector space

in other words, γ is parametrized by arc-length

Let γ_n be a sequence of immersed curves $\mathcal{C}^\infty$-approximating γ. We have $L(\gamma_n) \to 1$; however, we have no control on the periods of such γ_n, and



our formulas for A, L, E are for period 1 loops, so reparametrize:

$$\tilde{\gamma}_n : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}^n$$

Since $\tilde{\gamma}_n \xrightarrow{\mathcal{C}^\infty} \gamma$, we have, in particular:

$$\int_0^1 |\dot{\tilde{\gamma}}_n - \dot{\gamma}| \, dt = 0$$

$$\Rightarrow \lim_n |\dot{\tilde{\gamma}}_n - \dot{\gamma}| = 0 \quad \text{a.e}$$

Reverse triang. ineq.: $\left| |\dot{\tilde{\gamma}}_n| - |\dot{\gamma}| \right| \leq |\dot{\tilde{\gamma}}_n - \dot{\gamma}|$

So we have $|\dot{\tilde{\gamma}}_n| \longrightarrow |\dot{\gamma}| = 1$

In particular:

$$A(\tilde{\gamma}_n) \leq \frac{1}{2\pi} E(\tilde{\gamma}_n) = \frac{1}{4\pi} \int_0^1 |\dot{\tilde{\gamma}}_n| \qquad (1)$$

$n \to \infty$: $A(\tilde{\gamma}_n) \to A(\gamma)$ and $\frac{1}{4\pi} \int_0^1 |\dot{\tilde{\gamma}}_n| \to \frac{1}{4\pi}$

So $A(\gamma) \leq \frac{1}{4\pi} = \frac{1}{4\pi} L(\gamma)^2$

true since $L(\gamma) = 1$ in our case.
but why the ^{2} factor?

→ Let $\gamma^c = c\gamma$ for $c \in \mathbb{R}_+$
(i.e, a curve of length c).

Then, the equality (1) becomes

$$|A(\tilde{\gamma}_n^c)| \leq \frac{1}{2\pi} E(\tilde{\gamma}_n^c) = \int_0^1 |\dot{\tilde{\gamma}}_n^c|^2$$

but since we can choose these curves $\tilde{\gamma}_n^c$ to simply be re-scalings of our previous $\tilde{\gamma}_n$, we get:

$$\int_0^1 |\dot{\tilde{\gamma}}_n^c|^2 = \int_0^1 c^2 |\dot{\tilde{\gamma}}_n|^2 = c^2$$

So that we do indeed get $|A(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2$

$\square$

3) It remains to show:

$$\frac{1}{4\pi} L(\gamma)^2 \leq \frac{1}{2\pi} E(\gamma)$$

$$\Longleftrightarrow \frac{1}{2} L(\gamma)^2 \leq E(\gamma)$$

$$\Longleftrightarrow \frac{1}{2}\left(\int_0^1 |\dot{\gamma}|\right)^2 \leq \frac{1}{2}\int |\dot{\gamma}|^2$$

$$\Longleftrightarrow \left(\int_0^1 |\dot{\gamma}|\right)^2 \leq \int_0^1 |\dot{\gamma}|^2$$

follows from standard analysis (eg. Hölder's inequality)

In conclusion:

$$|A(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2 \leq \frac{1}{2\pi} E(\gamma)$$

Ex. 3

$\mathcal{P} = \mathcal{C}^\infty(S^1, M)$. x_s: path in $\mathcal{P}$ (so a path of loops in M)

Let $\tau := \frac{d}{ds}\Big|_{s=0} x_s$

(vector field along the loop $x_0 = x$)

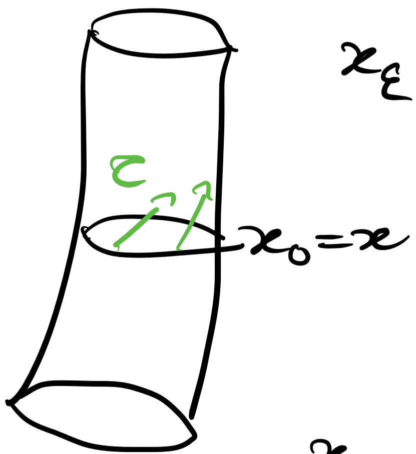
1. Let $\mathcal{A}_H : \mathcal{P} \to \mathbb{R} : x \longmapsto -\int_{S^1} x^*\lambda + \int_{S^1} H \circ x$

Then, for $\tau \in T_x\mathcal{P}$:

$$d\mathcal{A}_H(x)\tau = \frac{d}{ds} \mathcal{A}_H(x_s)\Big|_{s=0}$$

$$= -\frac{d}{ds}\int_{S^1} x_s^* \lambda + \underbrace{\int_{S^1} \frac{d}{ds} H \circ x_s}_{= \int_{S^1} dH(\tau)}$$

2.



$x : (-\varepsilon, \varepsilon) \to M$

our path on loops

$(\forall s : x_s : S^1 \to M)$

And define $\tau_s := \frac{d}{ds} x_s$

(so $\tau_0 = \tau$)

Then, $\forall \theta \in S^1$, the trajectory $s \mapsto x_s(\theta)$ can be viewed as a flow line of τ_s (since $\frac{d}{ds} x_s = \tau_s$)

And so there is a flow $\Psi_s : Z \to Z$ of τ_s, where Z is the cylinder

$$Z = \mathrm{im}\left\{ x_s(t) \;\middle|\; \begin{matrix} -\varepsilon \leq s \leq \varepsilon \\ t \in S^1 \end{matrix} \right\}$$

Note that, since $\forall \theta$, $\Psi^s(x_0(\theta)) = x_s(\theta)$ (by def^n)

we have $\boxed{\Psi^s \circ x_0 = x_s}$ (*)

RK. Now, to speak of the Lie derivative, one first needs to extend the flow of τ to a whole nhbd of $\mathrm{im}(x)$ in M. But this can be done w/o issue: just take any smooth extension of our τ_s on Z to a nhbd of it.

Then: $x^* \mathcal{L}_\tau \lambda = x_0^* \mathcal{L}_\tau \lambda$

$$= x_0^* \lim_{s \to 0} \frac{1}{s}(\Psi_s^* \lambda - \lambda)$$

$$= \lim_{s \to 0} \frac{1}{s}\left((\Psi_s \circ x_0)^* \lambda - x_0^* \lambda\right)$$

(*)

$$= \lim_{s \to 0} \frac{1}{s}(x_s^* \lambda - x_0^* \lambda)$$

$$= \frac{d}{ds}(x_s^* \lambda)\Big|_{s=0}$$ (by def^n of the derivative) $\square$

3. We have:

$$d\mathcal{A}_H(x)\,\tau = -\frac{d}{ds}\int_{S^1} x_s^*\lambda + \int_{S^1} dH(\tau)$$

by 2.

$$= -\int_{S^1} x^*\mathcal{L}_\tau\lambda + \int_{S^1} dH(\tau)$$

Now, by Cartan's magic formula:

$$\mathcal{L}_\tau\lambda = d(\iota_\tau\lambda) + i_\tau d\lambda$$

This is an exact 1-form, so its integral along S^1 is 0, by Stokes.

$$\therefore\ d\mathcal{A}_H(x)\,\tau = \int_{S^1} -x^*(i_\tau d\lambda) + dH(\tau)$$

since $i_{X_H}\omega = -dH$

$$= \int_{S^1} -x^*\left(d\lambda(\tau,\cdot)\right) - d\lambda(X_H,\tau)$$

for a 1-form η, $x^*\eta = \eta(\dot{x}(t))\,dt$

$$= -\int_{S^1}\left(d\lambda(\tau,\dot{x}(t)) + d\lambda(X_{H}(x(t)),\tau(t))\right)dt$$

$$= \int_{S^1} d\lambda(\dot{x}(t) - X_H(x(t)), \tau(t))\, dt$$

$\square$

4. Principle of least action:

Let $(M, \omega = d\lambda)$ be a compact exact symplectic manifold, and $H : M \to \mathbb{R}$ a Hamiltonian.

Then, periodic orbits of period 1 of H on M correspond to critical points of the action functional

$$\mathcal{A}_H : \mathcal{C}^\infty(S^1, M) \to \mathbb{R}$$
$$x \longmapsto -\int_{S^1} x^*\lambda + \int_{S^1} H \circ x$$

Pf: $x : S^1 \to M$ is a periodic orbit of the flow iff $\dot{x}(t) = X_H(x(t))$

iff $\left(d\mathcal{A}_H(x) = \int_{S^1} d\lambda(\dot{x} - X_H, \cdot) \right) \equiv 0$

$\square$

Exercise 4:

1. By defn, $\nabla \mathcal{A}_H$ must be s.t

$$\langle \nabla \mathcal{A}_H, \cdot \rangle = d\mathcal{A}_H(\cdot)$$

$$\langle \nabla \mathcal{A}_H, \cdot \rangle = -\int_{S^1} d\lambda(J\nabla\mathcal{A}_H, \cdot) \quad \text{by defn}$$

$$d\mathcal{A}_H(\cdot) = \int_{S^1} d\lambda(\dot{x}(t) - X_H, \cdot) \quad \text{by 3.}$$

Hence, $\nabla \mathcal{A}_H = J(\dot{x}(t) - X_H)$

Therefore, the equation

$$\frac{\partial u}{\partial s} = -\nabla \mathcal{A}_H(u(s))$$

reads

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H(u(s))\right) = 0$$

$X_H = J\nabla H$

$$\Longleftrightarrow \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H = 0 \qquad \square$$

2. $E(u)=0 \Leftrightarrow \frac{\partial u}{\partial s}=0$ a.e

$\Leftrightarrow \frac{\partial u}{\partial s}=0$ (by smoothness)

so u is constant in s

And $J\frac{\partial u}{\partial t}+\nabla H=0$

$\Leftrightarrow \frac{\partial u}{\partial t}=X_H(u(t))$

So u is actually a trajectory of the flow (a periodic orbit, since $t\in S^1$).

3. $$E(u)=\int_{\mathbb{R}\times S^1}|\partial_s u|^2\, ds\wedge dt$$

$$=\int_{\mathbb{R}\times S^1} g(\partial_s u,\partial_s u)\, ds\wedge dt$$

$$=\int_{\mathbb{R}\times S^1} -g(J(\partial_t u-X_H),\partial_s u)\, ds\wedge dt$$

$$= \int_{\mathbb{R}\times S^1} -\omega(\partial_t u - X_H, \partial_s u)\, ds \wedge dt$$

$$= \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_H)\, ds \wedge dt$$

$\square$

4. **Assume** $u : \mathbb{R}\times S^1$ is a cylinder s.t

$$\begin{cases} \lim_{s\to-\infty} u(s,\cdot) = x \\ \lim_{s\to\infty} u(s,\cdot) = y \end{cases}$$

periodic orbits of H.

Then note:

$$E(u) = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_H)\, ds \wedge dt$$

$$= \int_{\mathbb{R}\times S^1} d\lambda(\partial_s u, \partial_t u - X_H)\, ds \wedge dt$$

$$= -\int_{\mathbb{R}\times S^1} d\lambda(\dot{u} - X_H, \partial_s u)\, ds \wedge dt$$

$$= -\int_{\mathbb{R}} \left(\int_0^1 d\lambda(\dot{u} - X_H, \partial_s u)\, dt \right) ds$$

$$= -\int_{-\infty}^{\infty} d\mathcal{A}_H(\partial_s u)\, ds$$

$$= -\int_{-\infty}^{\infty} \frac{d}{ds}\left(\mathcal{A}_H(u(s))\right) ds$$

$$= \lim_{s \to -\infty} \mathcal{A}_H(u(s)) - \lim_{s \to \infty} \mathcal{A}_H(u(s))$$

$$= \mathcal{A}_H(x) - \mathcal{A}_H(y)$$

$\square$