**Ex 1.** Assume $\dim M = 3$.

Choose nhbd $\mathcal{U}$ with coords. $(x, y, z)$ and ctct form $\lambda = dz + x\,dy$

$$\left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 \leq r^2 \right\}$$

$$\Vert$$

Consider $\phi : B(r) \longrightarrow \mathcal{U} \hookrightarrow M$

$$(x,y,z) \longmapsto \left(\tfrac{1}{r}x, \tfrac{1}{r}y, \tfrac{1}{r^2}z\right)$$

$$\phi^* \lambda = r^2 (dz + x\,dy)$$

So $\phi : (B(r), dz + x\,dy) \hookrightarrow \left(\mathcal{U}, \frac{1}{r^2}\lambda\right)$

is a contact embedding.

This is what we wanted, since ex. was asking for $(B(r), \alpha_0) \hookrightarrow (M, \xi)$

contact structure $\xi = \ker \alpha$ is invariant up to rescaling the contact form $\alpha$.

$$\xi = \ker(\lambda) = \ker(c\lambda)$$

# Ex 2.

1. $q$: reg value of $f$

$\Rightarrow f^{-1}(q)$ : dim 0 submfd of $M$ (implicit function theorem)

(closed bc pre-image of a closed set by a cts function

$M$ cpct

$\Rightarrow f^{-1}(q)$ compact (as a closed subset of a cpct Hausdorff set)

$\Rightarrow \# f^{-1}(q) < \infty$

2. $\left.\begin{array}{l} q \text{ regular value} \\ \dim M = \dim N = n \end{array}\right\} \Rightarrow$ locally, $f$ has rank $n$

(ie $Df_q : T_qM \to T_{f(q)}N$ is an iso.)

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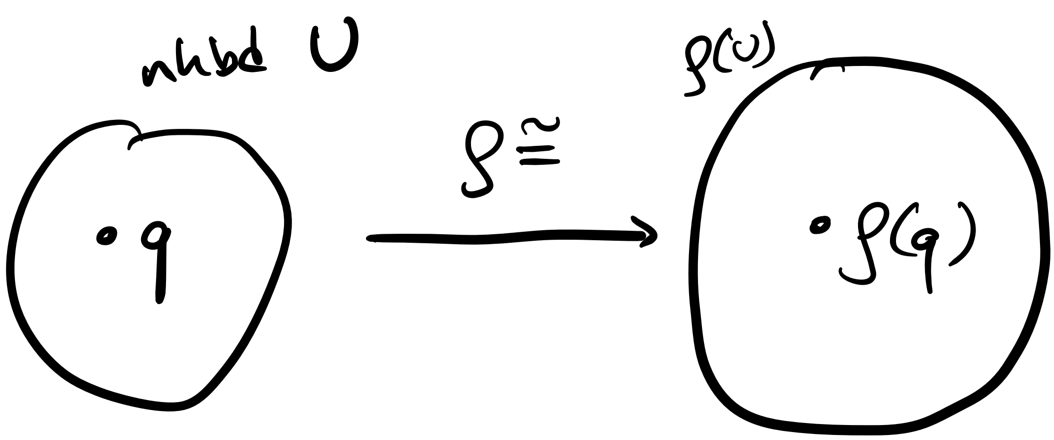
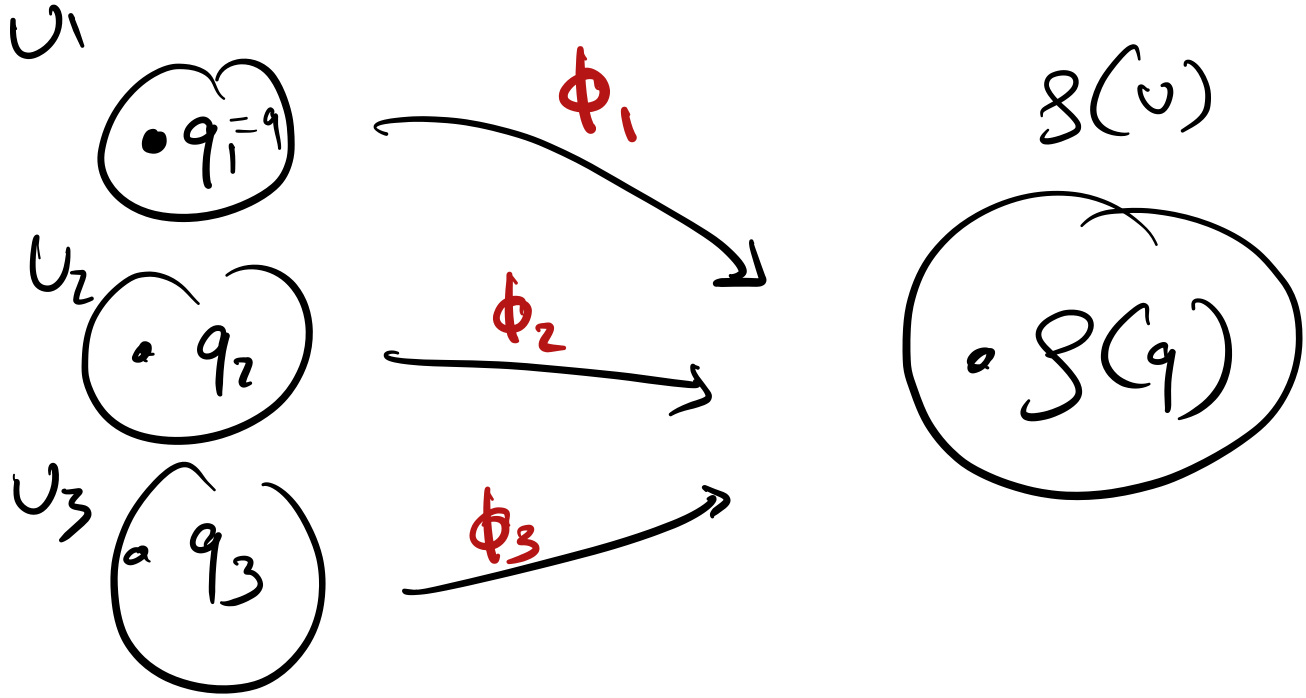
have a local diffeo



However, there might be multiple




wlog, $U_i$ are all disjoint

$$\left(\phi_i : U_i \xrightarrow{\cong} V_i \subseteq \rho(U)\right)$$

<u>W.T.S:</u> $q \xmapsto{\deg} \# \rho^{-1}(q)$ is loc. cst.

Let $q' \in V_1 \cap \ldots \cap V_k$ arbitrarily close to $q$

Then, if $\# \rho^{-1}(q') > k$, that means

$\exists\, p_{k+1} \notin U_1 \cup \dots \cup U_k$ s.t.

$$f(p_{k+1}) = q'$$

$$\Rightarrow q' = f(p_{k+1}) \in f(M \setminus U_1 \cup \dots \cup U_k)$$

$(*)$

So consider

$$V := V_1 \cup \dots \cup V_k \setminus f(M \setminus U_1 \cup \dots \cup U_k)$$

- open nhbd of $f(q)$ (by elementary topology)
- $V \subseteq R_f$ (by open-ness of the latter)

By $(*)$, $q' \in V$

$$\Rightarrow \nexists\, f^{-1}(q') = k \quad \square$$

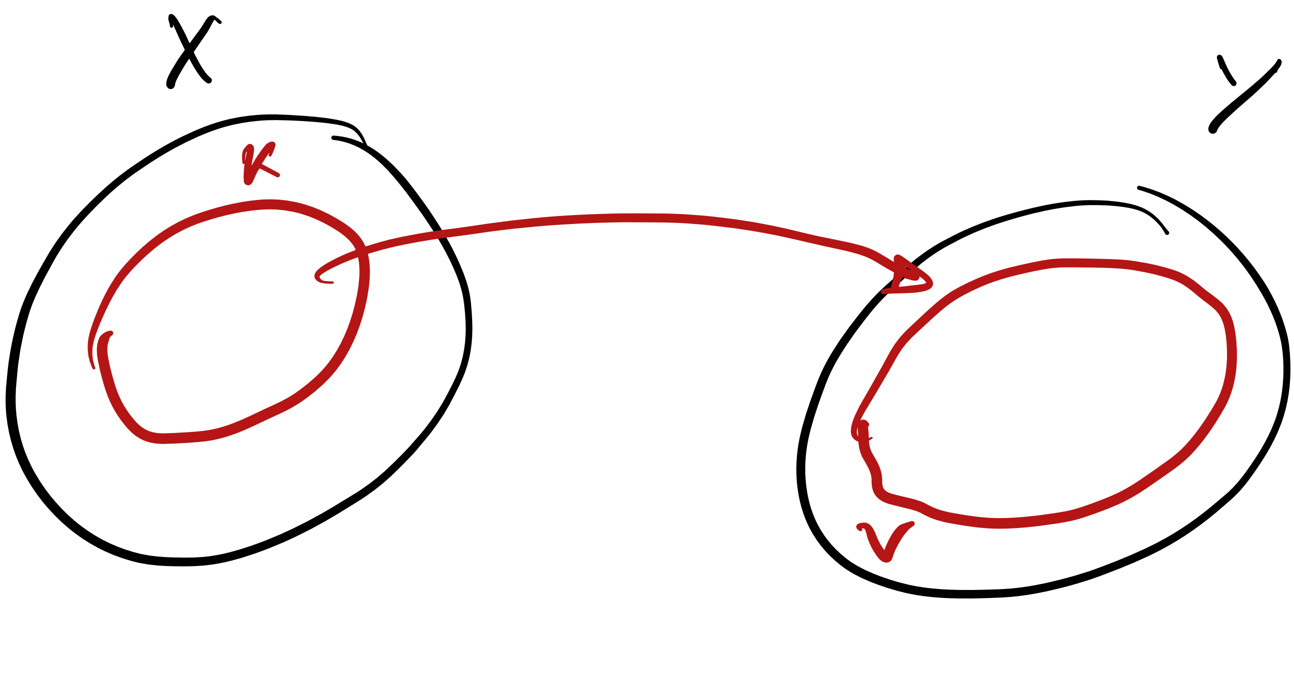
## Ex 3:

1. $X$ : compact
   $(Y, d_Y)$: metric space

$\vdash$ (Topologies are equivalent)

ie "$U$ open w.r.t the C-O topology" $\Leftrightarrow$ "$U$ is open w.r.t the top. induced by $d$"!

• ($\subseteq$) Let $f \in C(X, Y)$ be in the basic open set $U = S(K, V)$



<u>W.T.S</u> $\exists$ $\varepsilon > 0$ small enough,

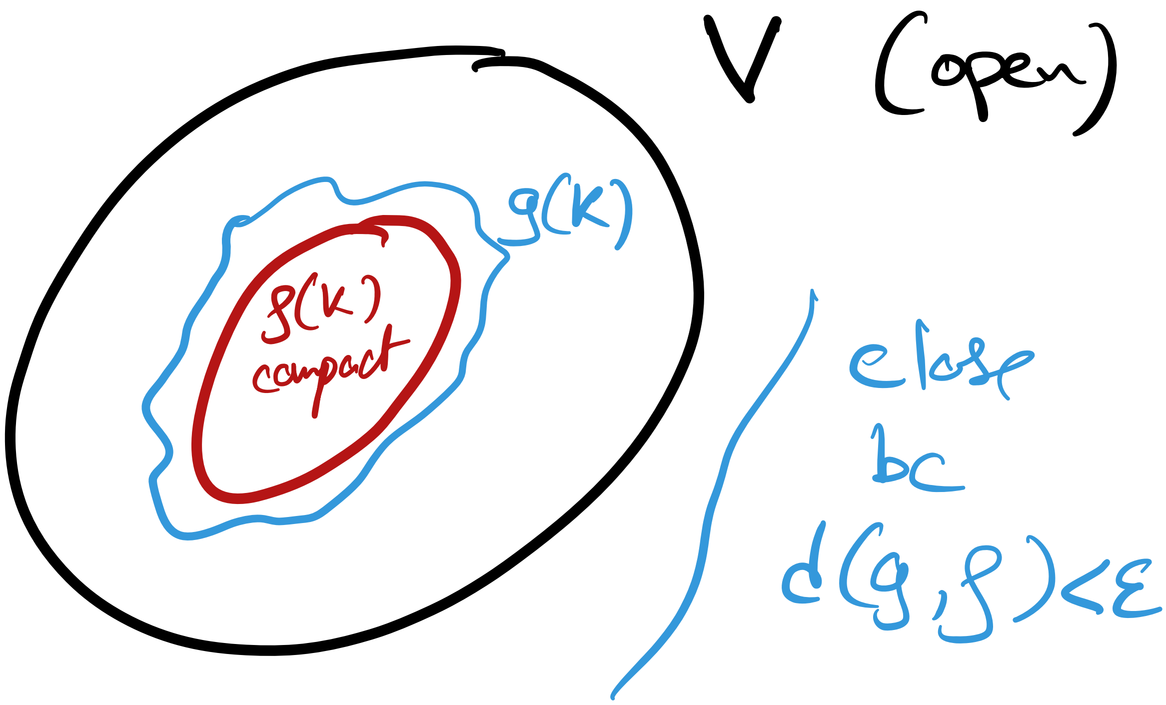
$$B_\varepsilon^{(f)} := \{ g \mid d(g,f) < \varepsilon \}$$

is contained in $S(K,V)$.

i.e that:

$$\left( d(g,f) < \varepsilon \implies g \in S(K,V) \right)$$

Visually:



For $\varepsilon > 0$ small enough,

$$g(K) \subseteq V$$

(this is bc $\underbrace{\operatorname{dist}(f(K), \partial V) = \delta > 0}$)

so can pick $\varepsilon := \frac{\delta}{2}$ to make sure $g(K) \subseteq V$

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(2) Now do the opposite.

Let $\underbrace{f \in C(X,Y)}$

with an open nbhd $B_r(f)$

Want to find $\underset{\substack{\cup\\ f}}{S(K,V)} \subseteq B_r(f)$

(ie, want to find $S(K,V) \ni f$

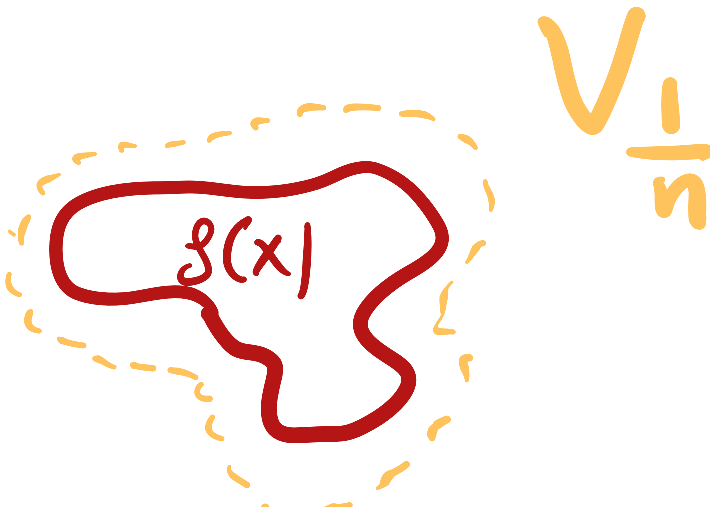
s.t. $d(f(x),g(x)) < r$

$\forall g \in S(K,V), \ \forall x \in K$)

Assume $\nexists$ such $S(K,V)$.

Then, $\exists$ sequence

$$S(K,V_{\frac{1}{n}}) \subseteq S(x,V_{\frac{1}{n}})$$




as well as $g_n \in S(x,V_{\frac{1}{n}}) \setminus B_r(f)$

meaning:

$$\begin{cases} d(g_n, f(x)) < \frac{1}{n} \\ \exists x_n \text{ s.t } d\big(g_n(x_n), f(x)\big) \geq r \end{cases}$$

<u>Contradiction !</u> $\square$

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2. $X = \bigcup_n K_n$ (wlog $K_n \Subset K_{n+1}$ else, set $A_n := \bigcup_{j \leq n} K_j$)

Define: $d(f,g) := \sum_n \frac{1}{2^n} \frac{d_K(f,g)}{1 + d_K(f,g)}$

($d_K :=$ previous dist. on $K$)

- This converges bc $0 < \frac{d_K}{1+d_K} < 1$

So sum $< \sum \frac{1}{2^n} < \infty$

$\rightsquigarrow$ props of a metric are ez to verify. $\square$

So this does indeed define a metric.

**Ex 4:** $Q(r) := (0,r)^{2n} \subseteq (\mathbb{R}^{2n}, \omega_0)$

$$\gamma(M,\omega) = \sup\{ r^2 \mid Q(r) \underset{\text{symp.}}{\hookrightarrow} M \}$$

- Monotone: ✓ (composition of symplectic embeddings is well-defined).

- Conformal: $\gamma(M, K\omega) = \underset{?}{\underbrace{?}} \, \gamma(M,\omega)$

want $|K|$

$(K>0)$

$$\begin{tikzcd}
& (M,\omega) \arrow[r, hook] & (M, K\omega) \\
& \vec{x} \arrow[r, mapsto] & \sqrt{K}\,\vec{x} \\
Q(r) \arrow[uur, hook] \arrow[rr, "\times\sqrt{K}"'] & & Q(\sqrt{K}\, r) \arrow[uu, hook]
\end{tikzcd}$$

- Note: as was rightly noticed by one of you in class, the diagram, written as such, only makes sense if $M \subseteq \mathbb{R}^{2n}$ (else, $\sqrt{K} M$ is ill-defined). However, in general, one can replace the top map by id, and instead use the $\sqrt{K}$ factor in the vertical inclusions.

So $\gamma(M, K\omega) \geq (\sqrt{K}\, r)^2 = K r^2$.

Diagram walks the other way,

so $\gamma(M, K\omega) = K r^2$.

If $\kappa < 0$, then do the same thing with an alternate embedding

$$(x_i, y_i) \longmapsto (\sqrt{\kappa}\, x_i, -\sqrt{\kappa}\, y_i)$$

So it's $\left\{ \begin{array}{l} \text{~~monotone~~} \\ \text{conformal} \end{array} \right.$ $\square$

• $\vdash \left( \gamma(Z(1), \omega_0) < \infty \right)$

Assume that $\forall r > 0$,

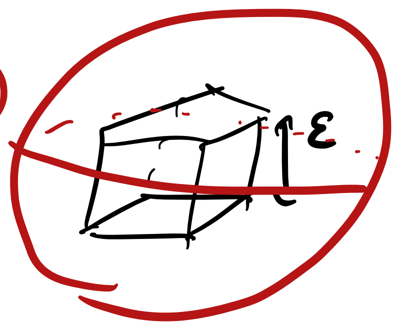
$$\begin{tikzcd}
\exists\, Q(r) \arrow[r, hook, "\text{symp.}"'] & Z(1) \\
B(r/2) \arrow[u, hook, "\text{symp.}"'] \arrow[ur] &
\end{tikzcd}$$

Contradicts Gromov's non-squeezing thm because that would mean we can symplectically embed balls of any size into $Z(1)$.

$\vdash \left(\gamma(B(1)) > 0\right)$

Find a cube which embeds into it.

B(1)




(Possible for $\varepsilon > 0$ small enough). □

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**Ex. 5** $h : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ homeo.

$$h^*\gamma = \gamma.$$

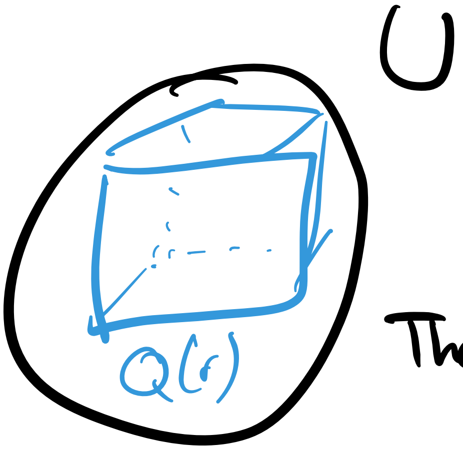
$\mu$: Lebesgue measure.

1. $\mu(Q(r)) = r^{2n}$ (it's just volume of the cube)

   Meanwhile, $\gamma(Q(r)) = r^2$

   So $\mu(Q(r)) = (\gamma(Q(r)))^n$ ✓

Now <u>W.T.S</u> $\left(\mu(U) \geq \gamma(U)^n\right)$




Let $r_* := \sup\{r \mid Q(r) \underset{s.}{\hookrightarrow} U\}$

Then $\mu(U) \underset{\text{monotonicity of Lebesgue measure}}{\geq} \mu(Q(r_*)) = r_*^{2n} = \gamma(U)^n$

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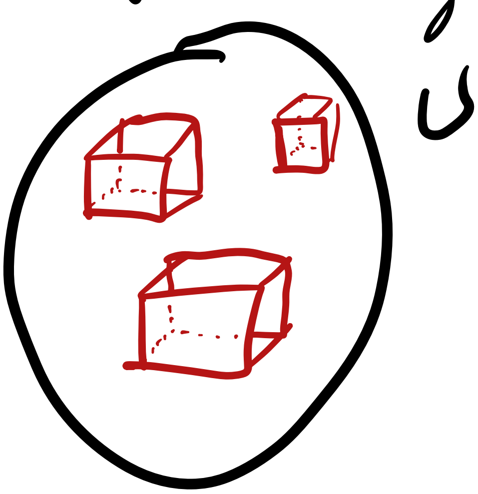
2\. $\mu(Q) \underset{\text{by !}}{=} \gamma(Q)^n \overset{\text{by assumption}}{=} \underbrace{\gamma(h(Q))^n \leq \mu(h(Q))}_{\text{by 1., using the fact } h(Q) \text{ is open}}$

3\. Lebesgue measure is <u>regular</u>

$\Rightarrow \forall \varepsilon > 0 \ \exists$ finitely many disj. cubes in $U$




such that:

$$\mu(U) - \varepsilon \leq \sum_i \mu(Q_i)$$

$$\leq \sum_i \mu(h(Q_i))$$

because the cubes are disjoint

$$= \mu\Big(h\big(\bigcup_i Q_i\big)\Big)$$

$$\leq \mu(h(U))$$

True $\forall \varepsilon > 0$ so have

$$\mu(U) \leq \mu(h(U))$$

4. $h$ is a homeomorphism so can re-do 1.–3. for $h^{-1}$. Get:

$$\begin{cases} \mu(U) \leq \mu(h(U)) \\ \mu(V) \leq \mu(h^{-1}(V)) \end{cases} \quad \forall \text{ open sets } U, V$$

$\rightsquigarrow$ pick $V = h(U)$. $\blacksquare$