I

Exercise 1 We do it for $Z(r)$. The reasoning for $B(r)$ is identical.

We know that $c(Z(1)) = \pi$.

Consider $Z(r) = r^2 Z(1)$. and the map

$$Z(r) \xrightarrow{\varphi} Z(1)$$
$$x \longmapsto \frac{x}{r}$$

$$\varphi^*(r^2\omega_0) = r^2 \varphi^*\omega_0 = r^2 \frac{1}{r^2}\omega_0 = \omega_0$$

$\Rightarrow (Z(r), \omega_0) \xrightarrow{\varphi} (Z(1), r^2\omega_0)$ is symplectic.

$$\Rightarrow c(Z(r), \omega_0) = c(Z(1), r^2\omega_0) \underset{\text{conformality}}{=} r^2 c(Z(1), \omega_0) = \pi r^2.$$

Consider now $Z_\infty(r) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x_1^2 + x_2^2 \le r^2\}$.

and the map $\varphi: B_N \longrightarrow \mathbb{R}^{2n}$.

$$(x,y) \longmapsto \left(\frac{rx_1}{N}, \frac{rx_2}{N}, x_3, \dots, x_n, \frac{N}{r}y_1, \frac{N}{r}y_2, y_3, \dots, y_n\right),$$

where B_N is the ball of radius N centered at $0 \in \mathbb{R}^{2n}$.

φ is a symplectic map and it squeezes B_N into $Z_\infty(r)$

$$\Rightarrow c(Z_\infty(r)) \ge c(B_N) = \pi N^2 \xrightarrow[N\to\infty]{} \infty \Rightarrow c(Z_\infty(r)) = \infty.$$

Exercise 2. (i) Consider $\varphi: B(1) \times B(1) \longrightarrow B(\frac{1}{r}) \times B(r)$

$$(x,y) \longmapsto \left(\frac{x}{r}, ry\right)$$

φ is volume preserving (but not symplectic!)

(ii) Clearly, we can symplectically embed $B(r) \times B(\frac{1}{r}) \hookrightarrow B(r) \times \mathbb{R}^{2n-2} = Z(r)$

$$\Rightarrow c\left(B(r) \times B(\tfrac{1}{r}), \omega_0\right) \le c\left(Z(r), \omega_0\right) = \pi r^2 \xrightarrow[r\to 0]{} 0.$$

Thus shows that in dimension > 2, volume and capacities are in general different things.

Exercise 3. (i) We have to verify definiteness of d_H and the triangle inequality.

If $d_H(A,B) = 0$, then $\sup\limits_{x\in A} d(x,B) = \sup\limits_{y\in B} (A,y) = 0$.

II

$\Rightarrow d(x,B) = d(A,y) = 0 \quad \forall x\in A, y\in B$

$\Rightarrow \forall n \geq 0 \ \exists y_n \in B: d(x,y_n) < \frac{1}{n}$. $\Rightarrow (y_n) \subseteq \mathbb{R}^{2n}$ is Cauchy

$\Rightarrow y_n \longrightarrow \bar{y} \in B$ (B is closed).

and $d(x,\bar{y}) = 0$. In particular $x \in B$. This shows $A \subseteq B$.

Similarly, we get $B \subseteq A$. $\Rightarrow A = B$.

Let now A, B, C be closed subsets of $\mathbb{R}^{2n}$. We want to show that

$d_H(A,C) \leq d_H(A,B) + d_H(B,C)$, i.e.

$$\sup_{x\in A} d(x,B) + \sup_{z\in C} d(A,z) \leq \sup_{x\in A} d(x,B) + \sup_{y\in B} d(A,y) + \sup_{y\in B} d(y,C) + \sup_{z\in C} d(B,z).$$

Let $\epsilon > 0$

Pick $\bar{x}\in A$, $\bar{z}\in C$. $d(\bar{x},C) + d(A,\bar{z}) \geq \sup\limits_{x\in A} d(x,B) + \sup\limits_{z\in C} d(A,z) - \epsilon$.

$\Rightarrow d_H(A,C) \leq d(\bar{x},C) + d(A,\bar{z}) + \epsilon$

$\leq d(\bar{x},y) + d(y,z) + d(x,y) + d(y',\bar{z}) + \epsilon \quad \forall \underset{A}{x}, \underset{B}{y, y'}, z \in C$.

Pick $\bar{y} \in B$: $d(\bar{x},\bar{y}) \leq d(\bar{x},B) + \epsilon$

Pick $\bar{z} \in C$, $d(\bar{y},\bar{z}) \leq d(\bar{y},C) + \epsilon$.

Pick $\bar{y}' \in B$: $d(\bar{y}',\bar{z}) \leq d(B,\bar{z}) + \epsilon$

Pick $\tilde{x} \in A$: $d(\tilde{x},\bar{y}') \leq d(A,\bar{y}') + \epsilon$.

Putting everything together we get

$$d_H(A,C) \leq (d(\bar{x},B)+\epsilon) + (d(\bar{y},C)+\epsilon) + (d(A,\bar{y}')+\epsilon) + (d(B,\bar{z})+\epsilon) + \epsilon$$

$$\leq \sup_{x\in A} d(x,B) + \sup_{y\in B} d(y,C) + \sup_{y\in B} d(A,y) + \sup_{z\in C} d(B,z) + 5\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get the claim.

(ii) Let $A \subseteq \mathbb{R}^{2n}$ be compact and convex and suppose that $\mathrm{int}(A) = \emptyset$.

Claim: A is contained in some hyperplane $W \subseteq \mathbb{R}^{2n}$.

Consider $V := \mathrm{span}(A)$. We claim that we can find a basis for V made up of elements of A.

Consider the family $\mathcal{F} := \{ S \subseteq A \mid S \text{ is linearly independent} \}$. (III)

$\Rightarrow$ $\mathcal{F}$ is a partially ordered set under inclusion, and it is nonempty.

Consider a chain $(S_i)_{i \in I} \subseteq \mathcal{F}$. Define $S := \bigcup_{i \in I} S_i$.

Then, S is linearly independent, as one can easily show.

$\rightsquigarrow$ by Zorn's lemma, $\mathcal{F}$ admits a maximal element. We will call it S.

S has to be a basis for V. Indeed, suppose $\mathrm{span}(S) \subsetneq V$.

Then, $\exists\, v \in V \setminus \mathrm{span}(S)$. Write $v = \sum_{i=1}^{n} \lambda_i a_i$, $a_i \in A$.

$\Rightarrow \exists\, i \in \{1, \dots, n\} \mid \{a_i, S\}$ is linearly independent, contradicting maximality of S.

If $\# S = 2n$, then convexity of A forces $\mathrm{int}(A) \neq \emptyset$.
(see picture below)

the shaded area is contained in A by convexity.

$\Rightarrow \# S < 2n$ and, by convexity, $A \subseteq W$ for some hyperplane ($\mathrm{span}(S)$ for example)

Write now $\mathbb{R}^{2n} = \langle v \rangle \oplus W$ for some $v \in \mathbb{R}^{2n}$.
$\exists\, w \in W : \omega(w, u) = 0 \ \forall u \in W$ (W is odd-dimensional). This forces $\omega(v, w) \neq 0$ and we can assume $\omega(v, w) = 1$. Extend v, w to a symplectic basis $v, w, w_3, \dots, w_{2n}$ for $\mathbb{R}^{2n}$, where $w_3, \dots, w_{2n} \in W$.

Consider $\mathbb{R}^{2n}$ with its standard symplectic coordinates $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ and symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Define the map $\mathbb{R}^{2n} = \langle v \rangle \oplus W \xrightarrow{\psi} \mathbb{R}^{2n}$

$$\begin{aligned} v &\longmapsto \tfrac{1}{\delta} e_1 \\ w &\longmapsto \delta e_2 \\ w_i &\longmapsto e_i \quad , i \geq 3. \end{aligned}$$

Then, $\varphi \in Sp(2n)$ and it is easy to see that $\varphi(A) \subseteq Z(\delta \cdot \mathrm{diam}(A))$.

This shows that $c(A) = c(\varphi(A)) \leq c(Z(\delta \cdot \mathrm{diam}(A)))$ $\downarrow \delta \to 0$ 0

$\longrightarrow c(A) = 0$.

IV

Now, if $\varphi(A) \subseteq Z(\delta)$ for some $\delta > 0$

and $d_H(A,B) < \delta$, then it is easy to see that $d_H(\varphi(A), \varphi(B)) < C\delta$

for some constant $C > 0$ not depending on δ (here, we are using the fact that φ is Lipschitz, as it is linear).

$\Rightarrow \exists\, C' > 0 : \varphi(B) \subseteq Z(C'\delta) \Rightarrow |c(B) - c(A)| = c(B) \xrightarrow{\delta \to 0} 0$.

This proves continuity at A in the case $\mathrm{int}(A) = \emptyset$.

Let us now consider the case $\mathrm{int}(A) \neq \emptyset$. Without loss of generality, suppose the unit ball $B_1(0)$ is contained in A.

Claim ∂A is homeomorphic to $\partial B_1(0) = S^{2n-1}$.

Let $v \in S^{2n-1} = \partial B_1(0)$. By convexity, $\exists!\, \mu > 0 : \mu v \in \partial A$.

Thus, the continuous map $\mathbb{R}^{2n} \setminus \{0\} \to S^{2n-1}$ restricts to a continuous
$v \longmapsto \frac{v}{\|v\|}$
bijection $\partial A \to S^{2n-1}$.

This is automatically a homeomorphism.

Claim $a \in A \Rightarrow a \in \mathrm{int}(\lambda A)$ $\forall \lambda > 1$.

This follows straight from the previous claim.

Claim $\forall \lambda > 1$ $\exists \delta > 0 : d_H(A,B) < \delta \Rightarrow \frac{1}{\lambda} A \subseteq B \subseteq \lambda A$.

We will show the inclusion $B \subseteq \lambda A$. The other one is identical.

Suppose the claim does not hold. Then, $\forall n \geq 0$ $\exists B_n$ cpt convex such that

$$\begin{cases} d_H(A,B) < \frac{1}{n} \\ B_n \not\subseteq \lambda A, \text{ i.e. } \exists\, b_n \in B_n : b_n \notin \lambda A. \end{cases}$$

The first condition implies that $\exists\, a_n \in A : d(a_n, b_n) < \frac{1}{n}$ $\forall n > 0$.

By compactness, we can assume $a_n \to \bar{a} \in A \Rightarrow b_n \to \bar{a}$.

In particular, b_n is arbitrarily close to $\bar{a}$.

By the previous claim, $\bar{c} \in \mathrm{int}(\lambda A)$. This gives a contradiction to the fact that $\forall x$.

Thus, the claim holds. V.

Now, we have $\forall \lambda > 1 \ \exists \delta > 0 : \ d_H(A,B) < \delta \Rightarrow \frac{1}{\lambda} A \subseteq B \subseteq \lambda A$.

$$\left(\frac{1}{\lambda}\right)^2 c(A) = c\left(\frac{1}{\lambda} A\right) \leq c(B) \leq c(\lambda A) = \lambda^2 c(A).$$

This proves continuity at A.