Exercise 1. Let (M,ω) be a compact symplectic manifold with $H^1_{\mathrm{dR}}(M) = 0$ (i.e all closed 1-forms are exact), and which we embed in $T^\star M$ as the zero section.

1. Let σ be a 1-form on M. Recall under which conditions $\mathrm{Graph}(\sigma)$ is Lagrangian in $T^\star M$.
2. Show that if σ is sufficiently $\mathcal{C}^1$ close close to zero, then $\mathrm{Graph}(\sigma)$ intersects the zero section in $T^\star M$ at least twice.

1. $\mathrm{Graph}(\sigma)$ is Lagr. iff σ is closed.

(from lectures)

Note: as pointed out by a question in class, the reasoning below only uses $\mathcal{C}^0$-closeness. In general, need $\mathcal{C}^1$ because we need to check that σ is a section (i.e $\pi\circ\sigma = \mathrm{id}$ for $\pi: T^*M \twoheadrightarrow M$; or more generally, that $\pi\circ\sigma : M \to M$ is a diffeomorphism)

↳ that's where we make use of the $\mathcal{C}^1$ assumption. Indeed,

$(\|\sigma - \mathrm{id}\|_{\mathcal{C}^1}$ sufficiently small$)$

$\Downarrow$

$(d\sigma \neq 0)$

$\Downarrow$

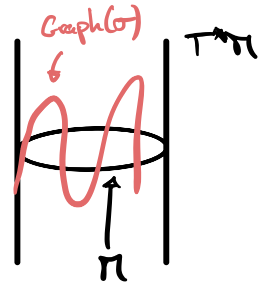
can use inverse fct thm

(See the next page for a written up proof. $(\ddagger)$)

2. $\sigma \underset{\mathcal{C}^1}{\sim} 0$

and $H^1_{dR}(M) = 0 \Rightarrow \exists f \in \mathcal{C}^\infty(M)$ s.t $\sigma = df$

$$\mathrm{Graph}(\sigma) = \{(p, \sigma(p)) \in T^*M\}$$



An intersection $\mathrm{Graph}(\sigma) \cap \{\text{zero section}\}$ is a point p s.t $\sigma(p) = 0$

$\Leftrightarrow df(p) = 0$

$\Leftrightarrow p$ is a crit. point of $f: M \to \mathbb{R}$.

M is cpt so there are at least two such crit. points. $\square$

(*) **Exercise** (i) Let $g : M \to T^*M$ be an embedding which is sufficiently close to the canonical embedding of the zero section in the C^1-topology. Prove that the image of g is the graph of a 1-form. (ii) Let $g : M \to M \times M$ be an embedding which is sufficiently close to the canonical embedding of the diagonal in the C^1-topology. Prove that the image of g is the graph of a diffeomorphism.

Solution (i) Let $z : M \to T^*M$ be the zero section embedding. We just need to show that if g is C^1-close to z then $\phi = \pi \circ g : M \to M$ is a diffeomorphism. Then letting $\sigma = g \circ \phi^{-1}$ we see that σ is an embedding (as a composition of an embedding and a diffeomorphism) and $\pi \circ \sigma = \pi \circ g \circ (\pi \circ g)^{-1} = \mathrm{id}$. Thus such a σ is a section with $\sigma(M) = g(M)$.

Assume that we have put a Riemannian metric g on M, thus inducing a metric (also g) on TM, T^*M and $T(T^*M)$ (the naturally induced metric on a TX and TX given a metric on X is easy to work out, but this is not the point of this question so we won't go into it here). Thus for two maps $\sigma, \tau : M \to T^*M$ and their corresponding differentials $d\sigma, d\tau : TM \to T(T^*M)$ we can define $\|\sigma - \tau\|_{C^0} = \max_{p\in M}\mathrm{dist}_g(\sigma(p), \tau(p))$ and $\|d\sigma - d\tau\|_{C^0} = \max_{(p,v)\in SM}\mathrm{dist}_g(d\sigma_p(v), d\tau_p(v))$ (here SM is the sphere bundle of TM under g), and thus $\|\sigma - \tau\|_{C^1} = \|\sigma - \tau\|_{C^0} + \|d\sigma - d\tau\|_{C^0}$.

Now consider the two maps $\phi = \pi \circ g$ and $i = \mathrm{id} = \pi \circ z$. We will start by showing that there is an $\epsilon_1 > 0$ such that $\|g - z\|_{C^1} < \epsilon_1$ implies that $d\phi : TM \to TM$ is rank n (i.e it's a local diffeomorphism).

Start by observing that the image $di(SM) = SM$. This is a compact sub-manifold of TM which is disjoint from the zero section $Z_0 \subset TM$. So the number $d(SM, M_0) = \min_{p\in M_0, q\in SM} d(p, q)$ is non-zero (it's 1 actually, assuming that we define the metric on TM in a reasonable way). Now, there exists a constant C_1 such that $\|d(\pi g) - d(\pi z)\|_{C^0} \le C_1\|g - z\|_{C^1}$ (this is evident since $\pi : TM \to M$ is C^∞ bounded and $d(\pi g) = d\pi \circ dg$). Now suppose that $\|g - z\|_{C^1} < \epsilon_1 = d(SM, M_0)/C_1$ and, for the sake of contradiction, that $dg_p(v) = 0$ for some $(p, v) \in SM$. Then we see that $d(dg_p(v), di_p(v)) = d((p, 0), (p, v)) > d(SM, M_0) = C_1\epsilon_1$. This contradicts the assumption that $\|d(\pi g) - d(\pi z)\|_{C^0} \le C_1\|g - z\|_{C^1} = C_1\epsilon_1$. Thus dg_p is non-degenerate (rank n) for each p in this case.

Now assume M is connected (the not connected case is just more notationally complicated but it isn't harder). The above argument shows that assuming $\|g - z\|_{C^1} < \epsilon_1$ implies that $\phi : M \to M$ is a covering map (we can show surjectivity using a continuity argument on M if it's connected). The fiber must be finite since M is compact. But the size of the fiber $|\phi^{-1}(p)|$ is locally constant near points p where $dg(p)$ is non-degenerate, and thus it is constant on M.Then the size of the fiber of g is some integer $n \ge 1$. We see that the fiber can be expressed as $F(\phi) = \int_M \phi^*\mu$ where μ is some fixed volume form with $\int_M \mu = 1$. But the map $F : C^\infty(M, M) \to \mathbb{R}$ given by this integral is certainly continuous in the C^1 topology, so for small ϵ_2 we must have $\|\phi - i\|_{C^1} < C_1\|g - z\|_{C^1} \le C_1\epsilon_2$ implies $F(\phi) = 1$ and thus that ϕ is a diffeomorphism.

Thus picking $\epsilon = \min(\epsilon_1, \epsilon_2)$ we see that $\|g - z\|_{C^1} < \epsilon$ implies that g is the graph of a section.

(ii) This admits a similar treatment to (i). Let $\delta : M \to M \times M$ denote the diagonal imbedding, and let $\pi_1, \pi_2 : M \times M \to M$ denote the two projection maps to the different factors. We want to show that if g is C^1-close enough to δ, then it is the graph of some diffeomorphism. It suffices to show that if g is close to δ

(**) in pink are the parts where we use the ϵ' assumption.

Exercise 2. Let (M, ω) be a compact symplectic manifold with $H^1_{\mathrm{dr}}(M) = 0$ and $f : M \to M$ a symplectomorphism.

1. Show that $\mathrm{Graph}(f)$ is Lagrangian $(M \times M, \omega \ominus \omega)$, where $\omega \ominus \omega := (\omega, -\omega)$.
2. Provided that f is sufficiently $\mathcal{C}^1$ close to the identity, explain how one can identify $\mathrm{Graph}(f) \subset M \times M$ with $\mathrm{Graph}(\eta) \subset T^\star M$ for some closed 1-form η on M.
3. Deduce that if f is a symplectomorphism which is sufficiently $\mathcal{C}^1$ close to the identity, then it has at least two fixed points.

This result can be refined by working on specific manifolds. For example, if $M = \mathbb{S}^2$, then *every* symplectomorphism has at least two fixed points. And since $\dim \mathbb{S}^2 = 2$, this can be rephrased as saying that every area-preserving diffeomorphism of $\mathbb{S}^2$ has at least two fixed points. And both of these conditions are essential!

4. Note that when we say "area-preserving", we really mean "area-form preserving"; so that our map not only preserves the absolute value of the area, but also the orientation. Show that, if we drop the second condition, then one can find diffeomorphisms of $\mathbb{S}^2$ with **zero** fixed point.
5. Find an example of diffeomorphism on $\mathbb{S}^2$ with exactly **one** fixed point.

   Hint: so you want your group to act freely on $\mathbb{S}^2 \backslash \{\mathrm{pt}\}$. What is this diffeomorphic to?

1. For simplicity, can first prove a linear version of this result:

$$\begin{cases} V: \text{ (finite-dim.) symplectic vector space} \\ A : V \to V \quad \text{endomorphism s.t. } A^*\omega = \omega \\ \tilde{\omega} := \omega \ominus \omega \quad \text{(2-form on } V \times V) \end{cases}$$

$\mathrm{Graph}(A) = \{(v, Av) \mid v \in V\} \subset V \times V$. Then:

$$\forall v, w \in V : \tilde{\omega}\big((v, Av), (w, Aw)\big) = \omega(v, w) - \underbrace{\omega(Av, Aw)}_{= A^*\omega = \omega}$$

$$= 0$$

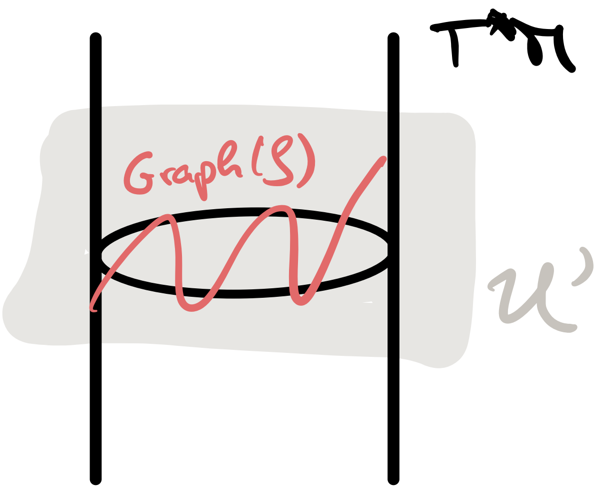
Hence, the statement is true for vector spaces; and a fortiori, locally for submfds (because it holds for their tangent spaces).

So, locally and every point in Graph(f), one can find a nhbd s.t $\tilde{\omega}|_{\mathrm{Graph}(f)} \equiv 0$. $\square$

2. Recall that $\Delta = \{(x,x) \mid x \in M\}$ is Lagrangian in $M \times M$. And by the Weinstein nhbd theorem, can find a nhbd $\mathcal{U}$ of Δ in $M \times M$, and map it to a nhbd $\mathcal{U}'$ of the zero section $(M \hookrightarrow T^*M)$ in T^*M.

If f is sufficiently $\mathcal{C}'$ close to id, then we can ensure that Graph(f) $\subset \mathcal{U}$ (recall M is cpct).

Hence, Graph(g) can be viewed as a Lagrangian in T^*M; which, moreover, is in a nhbd of the zero section.



$\longrightarrow$ if we find a 1-form σ on M s.t we can identify Graph(g) $\hookrightarrow M \times M$ with Graph(σ) $\hookrightarrow T^*M$, then by Ex. 1 σ is closed. So how to find such a σ?

"corresponds to" (after identification by Weinstein)

In T^*M, Graph(g) $\longleftrightarrow \left\{ (x, y_x) \,\middle|\, \begin{array}{l} x \in M \\ y \in T_x^*M \end{array} \right\}$

Since g is smooth (and so is our symplectomorphism), so is the map

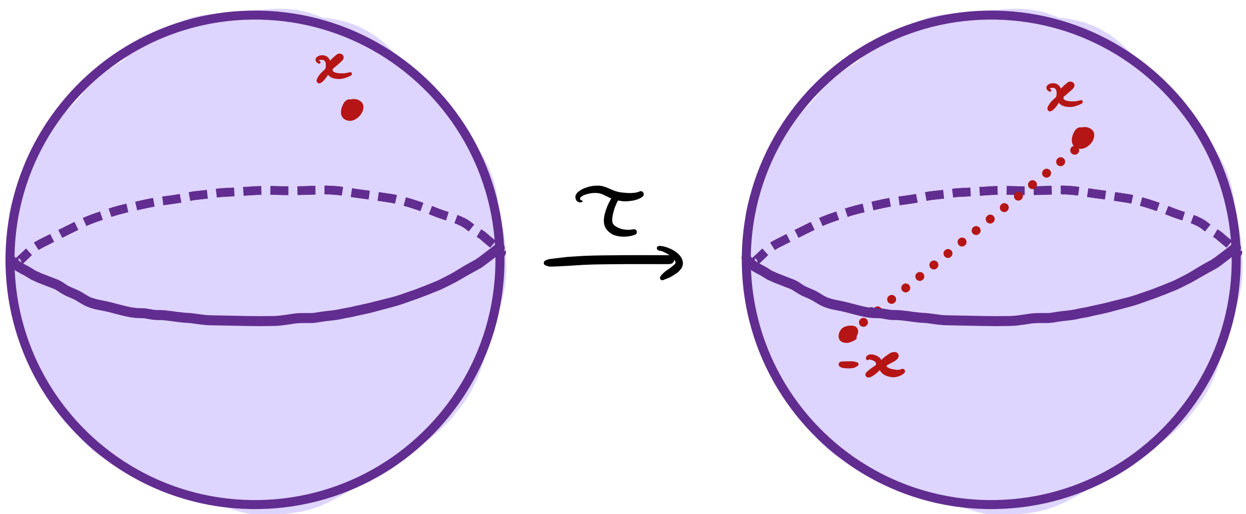
$$x \longmapsto y_x$$

which is, by def^n, a differential 1-form. $\square$

3. Directly follows from ex. 1.

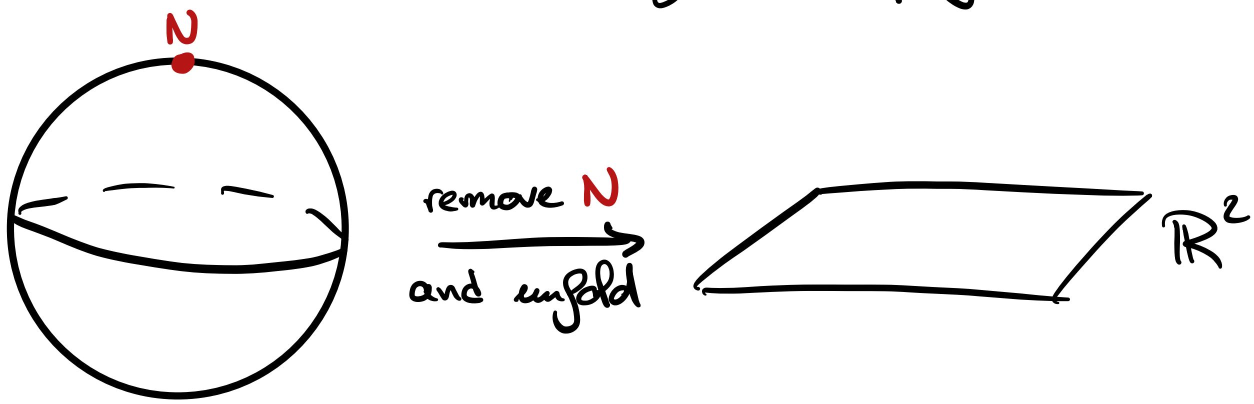
4. Consider the antipodal map

$$\tau : S^2 \to S^2 : x \longmapsto \underbrace{-x}_{\text{diametrically opposite to } x}$$



Then τ preserves area (up to sign), and is clearly a diffeomorphism; but it has no fixed points!

5. Consider the <u>stereographic projection</u>



Can now act on $\mathbb{R}^2$ by translations:

- fix $cst \in \mathbb{R}^2$
- let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : x \longmapsto x + cst$

pull this back along the stereographic projection (get $\tilde{\varphi} : S^2 \setminus \{N\} \to S^2 \setminus \{N\}$)

Can continuously extend it so that $\tilde{\varphi}(N) = N$

(follows from the fact that points at ∞ in $\mathbb{R}^2$ are mapped to points very close to N on S^2)

$\therefore$ $\tilde{\varphi}$ has only one fixed point.

(And one can easily show it's a diffeomorphism). $\square$

Exercise 3. Consider $\mathbb{R}^2$ with coordinates (q,p), and the action of $\mathbb{R}$ on $\mathbb{R}^2$ consisting of translation in the q-coordinate. Show that its moment map is given by p, the standard (linear) momentum from classical physics.

We consider the action $\mathbb{R} \circlearrowright \mathbb{R}^2$

$c \qquad (q,p) \mapsto (q+c, p)$

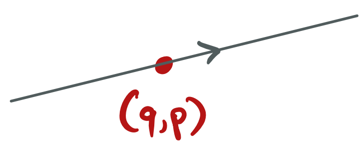
The infinitesimal generator for this action is

$$X_c = c\,\partial_q$$

Indeed, pick $(q,p) \in \mathbb{R}^2$, and $c \in \mathbb{R}$. The flow line through (q,p) is given by

$$\gamma(t) = (q + tc, p)$$

with derivative $\dot{\gamma}(t) = c\,\partial_q$.



$$\therefore \quad i_{X_c}\omega = i_{X_c}(dq \wedge dp)$$

$$= c\,dp$$

The (co-)moment map $\mu : \mathfrak{g} \longrightarrow C^\infty(M)$ is defined

s.t $\quad i_{X_c} = -d\mu(c)$

or $+$, depending on your sign convention.

So we must have $c\,dp = -d\mu(c)$

$$\Rightarrow \mu(c) = -cp$$

Then, the moment map is defined from the co-moment map by asking that:

co-moment map

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g}^* \\ p &\longmapsto \Big(c \longmapsto \underbrace{\mu_c(p)}_{= -cp} \Big) \end{aligned}$$

So $\mu_-(p)$ is the linear map

$$\mathbb{R} \longrightarrow \mathbb{R} : c \longmapsto -cp$$

(In our case, we can identify $\mathfrak{g} \cong \mathbb{R}$)

This is the same thing as the scalar $-p \in \mathbb{R}$.

$\therefore$ Moment map: $\mu : M \longrightarrow \mathfrak{g}^* ; p \longmapsto -p$

Note: most of the time, we rather consider

$\mu: p \mapsto p$ (for convenience); which is what you might have gotten with different sign conventions.

$\square$

Exercise 4. Let SO_3 denote the Lie group of rotations in $\mathbb{R}^3$, and recall that:

$$\mathfrak{so}_3 = \mathrm{Lie}(SO_3) = \{A \in \mathcal{M}_3(\mathbb{R}) \mid A + A^t = 0\}$$

1. Show that there is an isomorphism of Lie algebras $(\mathfrak{so}_3, [\cdot,\cdot]) \longrightarrow (\mathbb{R}^3, \times)$ given by:

$$\psi : \mathfrak{so}_3 \longrightarrow \mathbb{R}^3 : \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} (+) \longmapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

2. Compute the infinitesimal generator of the standard action of SO_3 on $\mathbb{R}^3$.

3. Deduce that $\mu = \vec{q} \times \vec{p}$ (the physical angular momentum) is a moment map for the action. not directly the action $SO_3 \curvearrowright \mathbb{R}^3$, but its cotangent lift $SO_3 \curvearrowright T^*\mathbb{R}^3$.

1. First, note that from the condition $A + A^t = 0$, every matrix $A \in \mathfrak{so}_3$ can be written in the form (+).

Clearly, $\psi : \mathfrak{so}_3 \to \mathbb{R}^3$ is an isomorphism of vector spaces; so need to check it preserves the Lie bracket.

⋮ calculation

(just do it for basis vectors of $\mathfrak{so}_3$)

✓

Bottomline:

$$\mathfrak{so}_3 \underset{\Psi^{-1}}{\overset{\Psi}{\rightleftarrows}} \mathbb{R}^3$$

$$\forall u, v \in \mathbb{R}^3, \quad [\Psi^{-1}u, \Psi^{-1}v] = \Psi^{-1}(u \times v)$$

---

will be useful later

Now notice: let $A \in \mathfrak{so}_3$ $\left(\text{so } A = \Psi^{-1}u \text{ for } u = (a_1, a_2, a_3)^t \right)$

$$A = \begin{pmatrix} & -a_3 & a_2 \\ a_3 & & a_1 \\ -a_2 & a_1 & \end{pmatrix}$$

- $Ae_1 = \begin{pmatrix} \\ a_3 \\ -a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ | $Ae_2 = \begin{pmatrix} -a_3 \\ \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ | $Ae_3 = \begin{pmatrix} a_2 \\ a_1 \\ \\ \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So in general, $\forall v \in \mathbb{R}^3$:

$$\boxed{Av = \Psi(A) \times v}$$

2. From now on, we are considering the cotangent lift to $T^*\mathbb{R}^3$ (i.e, the induced action $\mathfrak{so}_3 \circlearrowright T^*\mathbb{R}^3$).

↳ we're interested in the infinitesimal generator of this action. i.e:

For $A \in \mathfrak{so}_3$, $X_A = \frac{d}{dt}\Big|_{t=0} \Big(\exp(tA) \cdot (q,p) \Big)$

$$= \frac{d}{dt}\Big|_{t=0} \Big(\big(\mathbb{1} + tA + \frac{t^2}{2!}A + \dots\big) \cdot (q,p) \Big)$$

higher order terms disappear since we differentiate at $t=0$.

$$= \frac{d}{dt}\Big|_{t=0} \big(A \cdot (q,p) \big) = (Aq, Ap)$$

$\therefore$ $\boxed{X_A = (Aq, Ap)}$.

iii) Recall that by def^n, the co-moment map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathcal{C}^\infty(M) \\ A &\longmapsto \mu_A \end{aligned}$$

s.t $\iota_{X_A}\omega = + d\mu_A$

I chose the + sign convention. If you use −, you should get everything up to a sign.

And the moment map is defined as:

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g}^* \\ m &\longmapsto \left(\begin{aligned} \mathfrak{g} &\longrightarrow \mathbb{R} \\ A &\longmapsto \mu_A(m) \end{aligned} \right) \end{aligned}$$

Let's compute the first map first.

We have :

$$\iota_{X_A}\omega = (dq\wedge dp)(Aq, Ap)$$

$$= Aq\, dp - Ap\, dq$$

notice:

$$= d(p^t A q)$$

indeed:

$$d(p^t A q)$$

$$= (p^t A)\, dq + Aq\, dp^t$$

$$\overset{A^t = -A}{=} -(Ap)\, dq + Aq\, dp$$

same thing

Hence, the co-moment map is given by

$$\mu : \mathfrak{g} \longrightarrow C^\infty(M)$$

$$A \longmapsto p^t A q$$

Write $\langle \cdot , \cdot \rangle$ the std inner product.

$$\mu_A = p^t A q = \langle p, Aq \rangle$$

by 2.

$$= \langle p, \psi(A) \times q \rangle$$

And recall there is this cyclic equality for mixing cross & dot products:

$$\langle a, b\times c\rangle = \langle b, c\times a\rangle = \langle c, a\times b\rangle$$

Hence, $\mu_A = \langle \psi(A), q\times p\rangle$

And recall we want:

$$\begin{aligned} \mu: M &\longrightarrow \mathfrak{g}^* \\ m &\longmapsto \left(\begin{aligned} \mathfrak{g} &\longrightarrow \mathbb{R} \\ A &\longmapsto \mu_A(m) \end{aligned} \right) \end{aligned}$$

Can simply define $\mu(m) = \underbrace{q\times p \,|_m}$

indeed, this defines a vector in $\mathbb{R}^3$ (equivalently, a matrix in $\mathfrak{so}_3$; which we can identify with $\langle \cdot, q\times p\rangle \in \mathfrak{so}_3^*$.

So moment map: $\mu = q\times p$. $\square$