Exercise 1 Recall that the exponential map of matrix groups is given by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Consider $B = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and suppose $\exists A \in sp(2) = sl(2) : \exp(A) = B$

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $B \in sp(2) = sl(2)$, $tr(B) = 0 \Rightarrow d = -a$.

A small computation shows that $B^{2k} = (a^2+bc)^k I \quad \forall k \geq 0$

$$\Rightarrow \exp(B) = \sum_{k=0}^{\infty} \frac{B^{2k}}{(2k)!} + \sum_{\ell=0}^{\infty} \frac{B^{2\ell+1}}{(2\ell+1)!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(a^2+bc)^k}{(2k)!} \right) I + \left(\sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} \right) B$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(a^2+bc)^k}{(2k)!} + a \sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} & b \sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} \\ c \sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} & \sum_{k=0}^{\infty} \frac{(a^2+bc)^k}{(2k)!} - a \sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

If $\sum_{\ell=0}^{\infty} \frac{(a^2+bc)^\ell}{(2\ell+1)!} = 0 \Rightarrow$ the entries $(1,1)$ and $(2,2)$ are equal, but they should be equal to $-2, -\frac{1}{2}$ respectively.

$\Rightarrow b = c = 0 \Rightarrow \exp(B) = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$, which is also impossible, as $e^a > 0$.

Exercise 2. Write $V = \sum_{\lambda \in Sp(A)} E_\lambda$, where $E_\lambda = \ker(A - \lambda I)$ is the eigenspace corresponding to the eigenvalue λ. Recall that for symplectic matrices,

$$\lambda \in Sp(A) \Rightarrow \frac{1}{\lambda} \in Sp(A).$$

Using this, it is easy to see that E_1, E_{-1} are even-dimensional (they might be $\{0\}$).

Define $V_\lambda := E_\lambda \oplus E_{\frac{1}{\lambda}}$ if $\lambda \in Sp(A) \setminus \{\pm 1\}$, $|\lambda| \geq 1$.

If $e_1, \dots, e_{2n}$ is a basis of eigenvectors, we have

$\omega(e_i, e_j) = \omega(Ae_i, Ae_j) = \lambda_i \lambda_j \, \omega(e_i, e_j) \Rightarrow$ either $\lambda_i \lambda_j = 1$ or $\omega(e_i, e_j)$.

Thus, we get that $V_\lambda^\omega = \bigoplus_{\substack{\lambda' \neq \lambda \\ |\lambda'| \geq 1 \\ \lambda' \in Sp(A)}} V_{\lambda'}$. Indeed, the inclusion $(\supseteq)$ is clear by what we have just shown and dimension counting gives equality.

$\rightsquigarrow$ A splits as a direct sum of symplectic maps $\underset{A|_{V_\lambda}}{\overset{\shortparallel}{A_\lambda}} : V_\lambda \to V_\lambda$.

$\rightsquigarrow$ we can find a symplectic basis for each V_λ $\rightsquigarrow$ we can symplectically get A into block form.

$\rightarrow$ the only thing left to do is to symplectically diagonalize each A_λ.

If $\lambda = \pm 1$, $A_{\pm 1} = \pm I$ $\leftarrow$ it is already diagonal.

Suppose $\lambda \neq \pm 1$ and let $v \in E_\lambda$. By nondegeneracy of A_λ, $\exists\, \tilde{v} \in E_{\frac{1}{\lambda}}$ such that $\omega(v, \tilde{v}) = 1$. Write $V_\lambda = \langle v, \tilde{v} \rangle \oplus W$. Then, W is symplectic and

$W = (E_\lambda \cap \tilde{v}^\omega) \oplus (E_{\frac{1}{\lambda}} \cap v^\omega)$. We can keep going by recursion.

Exercise 3. (i) $\alpha_0 = dx_{2n+1} + \sum_{i=1}^{n} x_i \, dx_{i+n} \Rightarrow d\alpha_0 = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$.

$\Rightarrow \alpha_0 \wedge (d\alpha_0)^n = n! \, dx_1 \wedge dx_{n+1} \wedge \dots \wedge dx_n \wedge dx_{2n} \wedge dx_{2n+1}$

(ii) Let $\lambda : M \to \mathbb{R}$ be a nonvanishing function. Then, we compute

$d(\lambda\alpha) = d\lambda \wedge \alpha + \lambda \, d\alpha$

$\Rightarrow \lambda\alpha \wedge (d(\lambda\alpha))^n = \lambda\alpha \wedge \lambda^n d\alpha^n = \lambda^{n+1} \alpha \wedge (d\alpha)^n$, which is a volume form

$\Leftrightarrow \alpha \wedge (d\alpha)^n$ is.

(iii) Suppose $d\alpha|_{\xi = \ker\alpha}$ is degenerate at some point $p \in M$.

$\Rightarrow \exists \; v \in \xi_p \; : \; d\alpha_p(v, w) = 0 \quad \forall w \in \xi_p$.

Extend v to a basis $v, u_2, \dots, u_{2n}, \tilde{u}$ of T_pM, where $u_2, \dots, u_{2n} \in \xi_p$.

Then, we get $(\alpha \wedge (d\alpha)^n)(v, u_2, \dots, u_{2n}, \tilde{u}) = 0$, contradiction.

(iv) The existence of a degenerate direction follows from linear algebra: the nondegenerate part is always even-dimensional, but here M is $(2n+1)$-dimensional.

$\rightsquigarrow \exists \; R$ nonvanishing section such that $i_R d\alpha = 0$.

By what we have said, $R \notin \ker\alpha \Rightarrow \alpha(R) \neq 0$.

Define $R_\alpha := \dfrac{R}{\alpha(R)}$.

Exercise 4

(i) In general, given a skew-symmetric 2-form β on a vector space V, there is a canonical form. Let $U = \{ v \in V \mid \beta(v, \cdot) = 0 \} \Rightarrow$ there is a basis of V which looks like

$u_1, \dots, u_k, v_1, \dots, v_k, w_1, \dots, w_k$, where $u_1, \dots, u_k$ is a basis of U and

$$\begin{cases} \beta(v_i, v_j) = \beta(w_i, w_j) = 0 \\ \beta(v_i, w_j) = \delta_{ij}. \end{cases}$$

Now, $d\alpha|_0 : (T_0\mathbb{R}^{2n+1})^2 \to \mathbb{R}$ is a skew-symmetric 2-form, nondegenerate on $\ker\alpha_0 \subseteq T_0\mathbb{R}^{2n+1}$. Apply the above normal form to $d\alpha_0$ and declare the new coordinates to be the ones w.r.t. the basis given by the normal form Thm above.

$\Rightarrow d\alpha = \sum_{j=1}^{n} dx_j \wedge dy_j$

The kernel of $d\alpha$ (at $0 \in \mathbb{R}^{2n+1}$) is 1-dimensional and does not lie in $\ker \alpha_0$. ⇝ we can normalize the remaining coordinate to get $\alpha_0(\partial_z) = 1$.

(ii) $\alpha_t = (1-t)\alpha_0 + t\alpha$.

Note that at $0 \in \mathbb{R}^{2n+1}$,
$$\begin{cases} d\alpha_t = d\alpha_0 \\ \alpha_t = \alpha_0. \end{cases}$$

$\Rightarrow$ in a small enough nbd of 0, α_t is contact for all $t \in [0,1]$.

(iii) <u>Want:</u> find isotopy ψ_t of nbd of 0 such that $\psi_t^* \alpha_t = \alpha_0$.

⇝ $0 = \frac{d}{dt}(\psi_t^* \alpha_t) = \psi_t^*(\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t)$

⇝ want to solve $\mathcal{L}_{X_t}\alpha_t + \dot{\alpha}_t = 0$.

Write $X_t = H_t R_t + Y_t$, where
$$\begin{cases} R_t = \text{Reeb vector field of } \alpha_t \\ Y_t \in \xi_t = \ker \alpha_t. \end{cases}$$

$$\mathcal{L}_{X_t}\alpha_t = i_{X_t} d\alpha_t + d(\alpha_t(X_t)) = i_{Y_t} d\alpha_t + dH_t$$

⇝ we have to solve $\quad i_{Y_t} d\alpha_t + dH_t + \dot{\alpha}_t = 0 \qquad (*)$

Insert R_t ⇝ $dH_t(R_t) = -\dot{\alpha}_t(R_t)$. Choose a small enough nbd of $0 \in \mathbb{R}^{2n+1}$ so that R_t has no closed orbits ⇝ can integrate the above equation to get a smooth family H_t. Since $\dot{\alpha}_t = 0$ at $0 \in \mathbb{R}^{2n+1}$, we may require

$$\begin{cases} H_t(0) = 0 \\ d_0 H_t = 0. \end{cases}$$

Note that $dH_t + \dot{\alpha}_t$ can be seen as an element of ξ_t^*.
⇝ Y_t is uniquely determined by $(*)$ due to nondegeneracy of $d\alpha_t|_{\xi_t}$.

Note that $Y_t(0) = 0$ and hence $X_t(0) = 0$ ⇝ the flow of X_t fixes the origin. ⇝ integrate X_t to conclude the proof.