Exercise 1 ~~S symplectic implies $V = S \oplus S^\omega$.~~

(i) We will show the cyclic chain of implications

S symplectic $\Rightarrow$ S^ω symplectic $\Rightarrow$ $\omega|_S$ nondegenerate $\Rightarrow$ S symplectic.

- The first implication is true in light of $S^{\omega\omega} = S$.
- If S^ω is symplectic, we can write $V = S \oplus S^\omega$.

Let $u \in S$. $\omega(u,v) = 0 \quad \forall v \in S \Rightarrow \omega(u,\tilde{v}) = 0 \quad \forall \tilde{v} \in V$

$\Rightarrow u = 0$ by nondegeneracy of ω. $\Rightarrow \omega|_S$ is nondegenerate

Lastly, suppose S is not symplectic, i.e. $\exists u \neq 0$ in $S \cap S^\omega$.

$\Rightarrow \omega(u,v) = 0 \ \forall v \in S \Rightarrow \omega|_S$ is degenerate.

(ii) $(\Rightarrow)$ $S \subseteq S^\omega \Rightarrow \omega|_S = 0$.

$(\Leftarrow)$ $\omega|_S = 0$ ~~$\Rightarrow \omega(u,v) = 0$~~. Fix $u \in S$.

we have $\omega(u,v) = 0 \ \forall v \in S \Rightarrow u \in S^\omega$. This shows $S \subseteq S^\omega$.

(iii) This follows from the observation that for any two subspaces $X, Y \subseteq V$ such that $X \subseteq Y$, we have $Y^\omega \subseteq X^\omega$. This trivially holds.

(iv) $\dim V = \dim S + \dim S^\omega$.

$(\Rightarrow)$ S Lagrangian means $S = S^\omega \Rightarrow \begin{cases} \dim S = \frac{1}{2} \dim V \\ \omega|_S = 0 \text{ by (ii).} \end{cases}$

$(\Leftarrow)$ by (ii), S is isotropic, i.e. $S \subseteq S^\omega$. Since $\dim S = \dim S^\omega$, we get $S = S^\omega$.

Exercise 2 (i) $\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i$.

$\Rightarrow \omega_0^n = n!\, dq_1 \wedge dp_1 \wedge \dots \wedge dq_n \wedge dp_n$

$f^*\omega_0 = \omega_0 \Rightarrow f^*\omega_0^n = \omega_0^n$ (pullback commutes with wedge product).

(ii) $\left.\frac{d}{dt}\right|_0 \varphi^{t*}\omega_0 = \mathcal{L}_{X_H}\omega_0 = \underbrace{i_{X_H} d\omega_0}_{=0} + d\underbrace{i_{X_H}\omega}_{=dH} = d^2H = 0$.

$\Rightarrow \varphi^{t*}\omega_0$ is constant

But $\varphi^0 = id \Rightarrow \varphi^{t*}\omega_0 = \omega_0 \ \forall t$.

(iii) This is almost a tautology.

Exercise 3

(i) This is a local question, so we can work over $(\mathbb{R}^n_x \times \mathbb{R}^n_y, \omega_0 = \sum dx_i \wedge dy_i)$

α gets identified with a map $\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$, $x \longmapsto (x, f(x))$

$\Rightarrow$ locally, the tangent space to the graph of α is given by the span of the frame $\left\{ \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \right\}_{i=1,\dots,n} = \left\{ \frac{\partial}{\partial x_i} + \sum_j \frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial y_j} \right\}_{i=1,\dots,n}$

Thus, Graph(α) is Lagrangian $\iff \forall v, w \in \mathbb{R}^n$, we have

$$\omega_0\left(\sum_i v_i\left(\frac{\partial}{\partial x_i} + \sum_j \frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial y_j}\right), \sum_k w_k\left(\frac{\partial}{\partial x_k} + \sum_\ell \frac{\partial f_\ell}{\partial x_k}\frac{\partial}{\partial y_\ell}\right)\right) = 0$$

With $\omega_0 = \sum_s dx^s \wedge dy^s$, a quick computation gives

$$\sum_{i,j} (v_j w_i - v_i w_j)\frac{\partial f_j}{\partial x_i} = 0 \quad \forall v, w \in \mathbb{R}^n.$$

Choosing for instance $v = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we get $\sum_i w_i\left(\frac{\partial f_1}{\partial x_i} - \frac{\partial f_i}{\partial x_1}\right) = 0 \ \forall w$.

$\Rightarrow \frac{\partial f_1}{\partial x_i} - \frac{\partial f_i}{\partial x_1} = 0 \ \forall i$. Similarly, we get $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = 0 \ \forall i, j$.

This is precisely the condition $d\alpha = d\left(\sum f_i dx_i\right) = 0$.

(ii) We need to show that X_H is tangent to N.

If $p \in N$ and $v \in T_pN$, we have

$$\omega_p(X_H(p), v) = d_pH(v) = 0 \quad , \text{ since } v \in T_pN \subseteq T_p(H^{-1}(c)).$$

$\Rightarrow$ by maximality of dimension for Lagrangian subspaces, we get $X_H(p) \in T_pN$.

(iii) Define $\hat{\omega} = \omega + dh \wedge dt$.

$\Rightarrow (\hat{\omega})^{n+1} = \omega^n \wedge dh \wedge dt$ ✓ nowhere vanishing $\rightarrow$ $\hat{\omega}$ is symplectic.

Set $X_{\hat{H}} = Y + a\frac{\partial}{\partial h} + b\frac{\partial}{\partial t}$, where Y is tangent to M.

$$\Rightarrow i_{X_{\hat{H}}}\hat{\omega} = i_Y\omega + i_{a\frac{\partial}{\partial h} + b\frac{\partial}{\partial t}}(dh \wedge dt) = i_Y\omega + a\,dt - b\,dh$$

$$i_{X_{\hat{H}}}\hat{\omega} = d\hat{H} = dH_t + \frac{\partial H_t}{\partial t}dt - dh$$

$$\Rightarrow \begin{cases} Y = X_{H_t} \\ a = \frac{\partial H_t}{\partial t} \\ b = 1 \end{cases} \Rightarrow X_{\hat{H}} = X_{H_t} + \frac{\partial H_t}{\partial t}\frac{\partial}{\partial h} + \frac{\partial}{\partial t}.$$

(iv) $\alpha = \lambda + H dt$ $\Rightarrow$ $\alpha|_{\hat{N}} = \lambda|_{\hat{N}} + H|_{\hat{N}}\, dt|_{\hat{N}}$

$$= \lambda|_{\hat{N}} + h|_{\hat{N}}\, dt|_{\hat{N}}$$

- $\hat{\omega}|_{\hat{N}} = 0 \Leftrightarrow \omega|_{\hat{N}} + dh|_{\hat{N}} \wedge dt|_{\hat{N}} = d\lambda|_{\hat{N}} + d\left(h|_{\hat{N}}\, dt|_{\hat{N}}\right) = d\alpha|_{\hat{N}} = 0.$

(v) $\hat{N} \subseteq M \times \mathbb{R}^2$ Lagrangian.

$N_t \subseteq M \times \{t\} \times \mathbb{R}$

Then $\alpha|_{N_t} = \lambda|_{N_t} - H|_{N_t}\, dt|_{N_t} = \lambda|_{N_t}$

We know $\alpha|_{\hat{N}} = 0 \Rightarrow \alpha|_{N_t} = 0 \Rightarrow N_t$ is Lagrangian.

Exercise 4 (i) By the LES in homotopy of the fibration $SO_n \to SU_n \to SU_n/SO_n$

we get $\pi_1(SO_n) \to \underset{}{\overset{0}{\pi_1(SU_n)}} \to \pi_1(SU_n/SO_n) \to \pi_0(SO_n) \xrightarrow{\cong} \pi_0(SU_n)$

The last map is an iso and $\pi_1(SU_n) = 0$ by assumption $\Rightarrow \pi_1(SU_n/SO_n) = 0$.

Exercise 5

(i) $\det_{\mathbb{C}}^2 : U_n/O_n \longrightarrow S^1$ is clearly well-defined and it is a fibration. What is the fiber?

Let us rewrite U_n/O_n in a clearer form. There is a group isomorphism

$$\begin{aligned} SU_n \rtimes S^1 &\longrightarrow U_n \\ (n,h) &\longmapsto nh \end{aligned}$$

, where $SU_n \rtimes S^1$ is the semidirect product of SU_n and S^1 and we identify S^1 with the subgroup of U_n ~~[illegible]~~ given by

$$\left\{ \begin{pmatrix} z & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} \in U_n \,\middle|\, z \in S^1 \right\}.$$

Similarly, we can write $O_n \cong SO_n \rtimes \mathbb{Z}_2$, where we identify

$$\mathbb{Z}_2 \cong \left\{ \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} \in O_n \right\}$$

Then, it follows that $U_n/O_n \cong SU_n/SO_n \rtimes S^1/\mathbb{Z}_2$

This shows that the fiber of $\det_{\mathbb{C}}^2 : U_n/O_n \longrightarrow S^1$ is exactly SU_n/SO_n.

From the LES of the fibration $\det_{\mathbb{C}}^2$, we get

$$\underset{\substack{\| \\ 0}}{\pi_1(SU_n/SO_n)} \longrightarrow \pi_1\left(SU_n/SO_n \rtimes S^1/\mathbb{Z}_2\right) \xrightarrow{\pi_1(\det_{\mathbb{C}}^2)} \pi_1(S^1)$$

This shows that $\det_{\mathbb{C}}^2$ induces an injection on π_1-groups.

It also induces a surjection (this is clear) $\Rightarrow \det_{\mathbb{C}}^2 : U_n/O_n \longrightarrow S^1$ induces an isomorphism on π_1.