Problem Sheet #2 – Solutions

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Solutions

Exercise 1.

1. Riemannian metrics are nondegenerate. This means that the map

$$g^\sharp \colon TM \to T^*M : v \mapsto g(v, \cdot)$$

is a bundle isomorphism. This shows that ∇f is uniquely determined by the requirement $g(\nabla f, \cdot) = df$.

2. The Hamiltonian vector field X_H associated to a function $H \colon M \to \mathbb{R}$ is defined analogously, i.e. $i_{X_H}\omega = dH$. As above, this is well-defined due to nondegeneracy of ω.

3. This is a straightforward computation. The expression is

$$X_H = \sum_{i=0}^{n} \frac{\partial H}{\partial p_i}\partial_{q_i} - \sum_{i=0}^{n} \frac{\partial H}{\partial q_i}\partial_{p_i}.$$

4. From the previous point, the flow of X_H solves the system

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases}$$

for all $1 \leq i \leq n$. These are Hamilton's equations from classical physics.

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Exercise 2.

1. The potential energy is (up to constant) $V(q) = \frac{1}{2}kq^2$. From the previous exercise we get that $X_H = p\partial_q - kq\partial_p$.

2. This is a straighforward computation, using (3) of Problem 1.

3. We compute
$$\begin{aligned} X_H f = df(X_H) &= \omega_0(X_f, X_H) \\ &= \{f, H\}. \end{aligned}$$
This shows that $X_H f = 0 \iff \{f, H\} = 0$.

4. This is the same computation as above, the only difference being that now $X_H f$ does not compute the time derivative of f along the trajectories, since f is time-dependent by assumption. When differentiating with respect to time, the extra term $\partial_t f$ pops up.

Exercise 3.

1. ($\Rightarrow$) We only need to show that g_J is symmetric. For $v, w \in V$ we have
$$g_J(v, w) = \omega(v, Jw) = \omega(Jv, -w) = \omega(w, Jv) = g_J(w, v).$$
($\Leftarrow$) Follow the above argument backwards.

2. ($\Rightarrow$) We have for all $v, w \in V$
$$g_J(Jv, Jw) = \omega(Jv, J^2 w) = \omega(v, Jw) = g_J(v, w).$$
($\Leftarrow$) Follow the above argument backwards.

3. (1) $\implies$ (2) We argue by induction on $\dim V$. Let $u_1 \neq 0$ be a vector in V such that $g_J(u_1, u_1) = 1$. and consider the subspace $W \subseteq V$ spanned by u_1, Ju_1. Then, we have $V = W \oplus W^\omega$. Note that W^ω is J-invariant and that $\omega|_{W^\omega}$ is nondegenerate. Induction then finishes the proof.

   (2) $\implies$ (3) Define the vector space isomorphism $\phi\colon \mathbb{R}^{2n} \to V$ by sending $v_i \mapsto u_i$ and $w_i \mapsto J(u_i)$.

   (3) $\implies$ (1) It follows from the fact that J_0 is compatible with ω_0.

Exercise 4.

1. The symplectic form in the given basis is represented by the matrix $-J$. Thus, the condition $A \in \mathrm{Sp}(2n)$ reads $A^t(-J)A = -J$.

2. We can mimic the proof for $\mathrm{O}(n)$. Let $\mathfrak{so}(2n)$ be the space of skew-symmetric $2n \times 2n$ matrices and define
$$f\colon M_{2n}\mathbb{R} \to \mathfrak{so}(2n)\colon A \mapsto A^t JA.$$
Then, $\mathrm{Sp}(2n) = f^{-1}(J)$. As for $\mathrm{O}(n)$, it is easy to show that J is a regular value of f. Thus, $\mathrm{Sp}(2n)$ is a manifold. Moreover, it is clear that composition and inversion preserve $\mathrm{Sp}(2n)$.

3. The above costruction of the smooth structure on the symplectic group also gives us for free its Lie algebra, since

$$\begin{aligned}\mathfrak{sp}(2n) &= T_I \mathrm{Sp}(2n) \\ &= \ker d_I f = \{B \in M_{2n}\mathbb{R} \colon B^t J + JB = 0\}\end{aligned}$$

Alternatively, one can use the exponential map to construct curves in the symplectic group with specified initial derivative. Indeed, we can consider a matrix B such that $B^t J + JB = 0$ and define the curve

$$\gamma \colon \mathbb{R} \to \mathrm{Sp}(2n) \colon t \mapsto \exp(tB)$$

To show that the image of γ actually lies in the symplectic group, we compute

$$\begin{aligned}\gamma(t)^t J \gamma(t) &= \exp(tB)^t J \exp(tB) \\ &= \exp(tB)^t \exp\big(-tB^t\big) J = J,\end{aligned}$$

where in the second equality we used the Taylor expansion of the exponential map and the condition $B^t J + JB = 0$ to get

$$J \exp(tB) = \exp\big(-tB^t\big) J.$$

This shows that

$$\{B \in M_{2n}\mathbb{R} \colon B^t J + JB = 0\} \subseteq \mathfrak{sp}(2n).$$

The reverse inclusion is straightforward.