Ex. 1 1. (M,ω) Liouville domain

$\Rightarrow$ ω exact $(\omega = d\lambda)$

$\Rightarrow$ $\forall u : S^2 \to M$:

$$\int_{S^2} u^*\omega = \int_{\underbrace{\partial S^2}_{=\emptyset}} u^*\lambda = 0$$

So Liouville domain $\Rightarrow$ symp. aspherical

actually, exactness of ω is sufficient

2. Let $L \subseteq M$ exact (ie $\exists f \in C^\infty(L)$ s.t $\lambda|_L = df$), and $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$

Then:
$$\int_{\mathbb{D}^2} u^*\omega = \int_{\mathbb{D}^2} u^* d\lambda = \int_{\partial\mathbb{D}^2} u^*\lambda$$

$u(\partial\mathbb{D}^2) \subseteq L$

$$= \int_{\partial\mathbb{D}^2} u^* df$$

Stokes

$$= 0$$

$\square$

3.

The tangent bundle of R is given by:

$$TR := \left\{ (\xi_0, 0, \xi_2, v_0, 0, v_2) \mid \xi_0^2 + \xi_2^2 = 1, \begin{pmatrix} \xi_0 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_2 \end{pmatrix} = 0 \right\} \hookrightarrow T^\star \mathbb{S}^2$$

Thus, its normal bundle (in ambient space $T^\star\mathbb{S}^2$) is given by:

$$\begin{aligned} NR :&= \{ (\xi, v) = (\xi_0, 0, \xi_2, v_0, v_1, v_2) \mid \xi_0^2 + \xi_2^2 = 1, v \cdot w = 0 \ \forall w \in TM \} \\ &= \{ (\xi, v) = (\xi_0, 0, \xi_2, 0, v_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, v_1 \in \mathbb{R} \} \end{aligned}$$

The conormal bundle is then obtained by fibrewise dualizing NR, so that:

$$N^\star R = \{ (\xi_0, 0, \xi_2, 0, \eta_1, 0) \mid \xi_0^2 + \xi_2^2 = 1, \eta_1 \in \mathrm{End}(T_{(\xi_0,\xi_2)} R, \mathbb{R}) \}$$

But we can identify $\mathrm{End}(T_{(\xi_0,\xi_2)}R, \mathbb{R}) \cong \mathbb{R}$. Hence, we get:

$$\therefore\ N^* R \cap \mathbb{D}^* \mathbb{S}^2 \cong \left\{ (\xi_0, 0, \xi_2, 0, \eta_1, 0) \,\middle|\, \begin{array}{l} \xi_0^2 + \xi_2^2 = 1 \\ |\eta_1| \leq 1 \end{array} \right\}$$

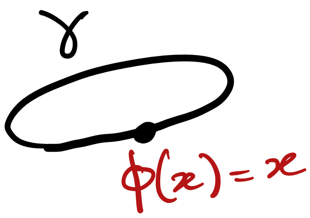
But the symp. structure on $\mathbb{D}^*\mathbb{S}^2$ stems from the one on $T^*\mathbb{S}^2$, namely

$$\omega = \sum_i d\xi_i \wedge d\eta_i = d\underbrace{\left(-\sum \eta_i d\xi_i\right)}_{\lambda}$$

Then, clearly λ vanishes on $N^*R \cap \mathbb{D}^*\mathbb{S}^2$, verifying that the latter is indeed an exact Lagr. (
- Lagrangian bc $\omega \equiv 0$, and it's of the right dimension.
- exact bc $\lambda \equiv 0$.

)

Ex 2. Ham. Floer theory detects 1-periodic orbits of $H: M \to \mathbb{R}$.



But $\left\{ \begin{matrix} \text{1-periodic} \\ \text{orbits of } H \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{fixed points} \\ \text{of } \underbrace{\phi: M \to M}_{\text{time-1 map of } H} \end{matrix} \right\}$

$\hookrightarrow$ As we already argued in Sheet #5, these are in bijection with intersections btwn Graph(ϕ) and Δ in $M \times M$.

both are Lagrangians (in a cpct, sympl, symp. aspherical mfd) hence studying their intersections is the point of Lagrangian Floer Homology.

Hence, Ham. Floer homology is a special case of Lagrangian Floer Homology.

Ex 3.

We want to show that $\mathrm{Graph}(\phi) \pitchfork \Delta \iff D\phi$ does not have 1 as an eigenvalue, where recall that $\phi = \phi_H^{t=1}$ is the Hamiltonian diffeomorphism of H.
Since both Lagrangians have dimension $\frac{1}{2}\dim M$, their intersection being transverse reduces to requiring that:

$$\forall p \in \mathrm{Graph}(\phi) \cap \Delta, \text{ we have: } T_p\mathrm{Graph}(\phi) \cap T_p\Delta = \emptyset$$

It is enough to work in a local neighbourhood, so let us identify M with $\mathbb{R}^{2n}$ and $M \times M$ with $\mathbb{R}^{4n}$. Then, $\mathrm{Graph}(\phi)$ is locally the zero set of the function:

$$\Xi : M \times M \longrightarrow M : (x, y) \longmapsto y - \phi(x)$$

which means that at a point $p \in \mathrm{Graph}(\phi)$, we have:

$$T_p\mathrm{Graph}(\phi) = \ker D\Xi|_p = \ker\left(-\mathrm{d}\phi \mid \mathrm{id}\right)|_p = \left\{ \begin{pmatrix} Z \\ \mathrm{d}\phi|_p Z \end{pmatrix} \mid Z \in \mathbb{R}^{2n} \right\}$$

While the diagonal Δ is the zero-set of

$$\Theta : M \times M \longrightarrow M : (x, y) \longmapsto y - x$$

which means its tangent space at a point p is:

$$T_p\Delta = \ker D\Theta|_p = \ker\left(\mathrm{id} \mid -\mathrm{id}\right) = \left\{ \begin{pmatrix} Z \\ Z \end{pmatrix} \mid Z \in \mathbb{R}^{2n} \right\}$$

Therefore, if $p \in \mathrm{Graph}(\phi) \cap \Delta$, then :

$$T_p\mathrm{Graph}(\phi) \cap T_p\Delta = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{4n} \mid Y = X = \mathrm{d}\phi|_p X \right\} \tag{C.1}$$

This shows that:

$$\begin{aligned} \text{The intersection is transverse} &\iff T_p\mathrm{Graph}(\phi) \cap T_p\Delta = \emptyset \\ &\iff \nexists X \in \mathbb{R}^{2n} \text{ s.t } X - \mathrm{d}\phi|_p X = 0 \\ &\iff 1 \text{ is not an eigenvalue of } \mathrm{d}\phi|_p \end{aligned}$$

□