# Symp. Geometry - Exercise class 1.

**Ex 2.** $M$: manifold.

1. $\Omega^k(M) := \{ \text{differential } k\text{-forms on } M \}$

Recall (ex. 1): linear form: multilinear & alternating map on a vector space

In general

$$\begin{array}{c} TM \text{: tangent bundle} \\ \downarrow \\ M \end{array} \qquad TM = \bigsqcup_{pt} \underbrace{T_{pt}M}_{\text{vector space}}$$

A differential form $\omega$: at every $pt \in M$,

$$\underbrace{\omega|_{T_{pt}M} \text{ : is a linear form}}_{\text{(glue that together smoothly)}}$$

$\rightsquigarrow$ Differential form: section of

$$\begin{array}{c} \Lambda^k T^*M \\ \downarrow \\ M \end{array}$$

- $\Lambda^k M = \left\{ \begin{array}{l} \text{objects } \omega \text{ s.t.} \\ \omega|_{T_{pt}M} \text{ is a linear form} \end{array} \right\}$

2. $d: \Omega^k(M) \to \Omega^{k+1}(M)$

$$\omega = \sum_{i_1 < \dots < i_k} f_{i_1,\dots,i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$\downarrow$$

$$d\omega := \sum_j \sum_{i_1<\dots<i_k} \frac{\partial f_{i_1,\dots,i_k}}{\partial x_j} \wedge dx_j \wedge dx_{i_1} \wedge \dots$$

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3. $M$: mfd, compact, orientable (no boundary)

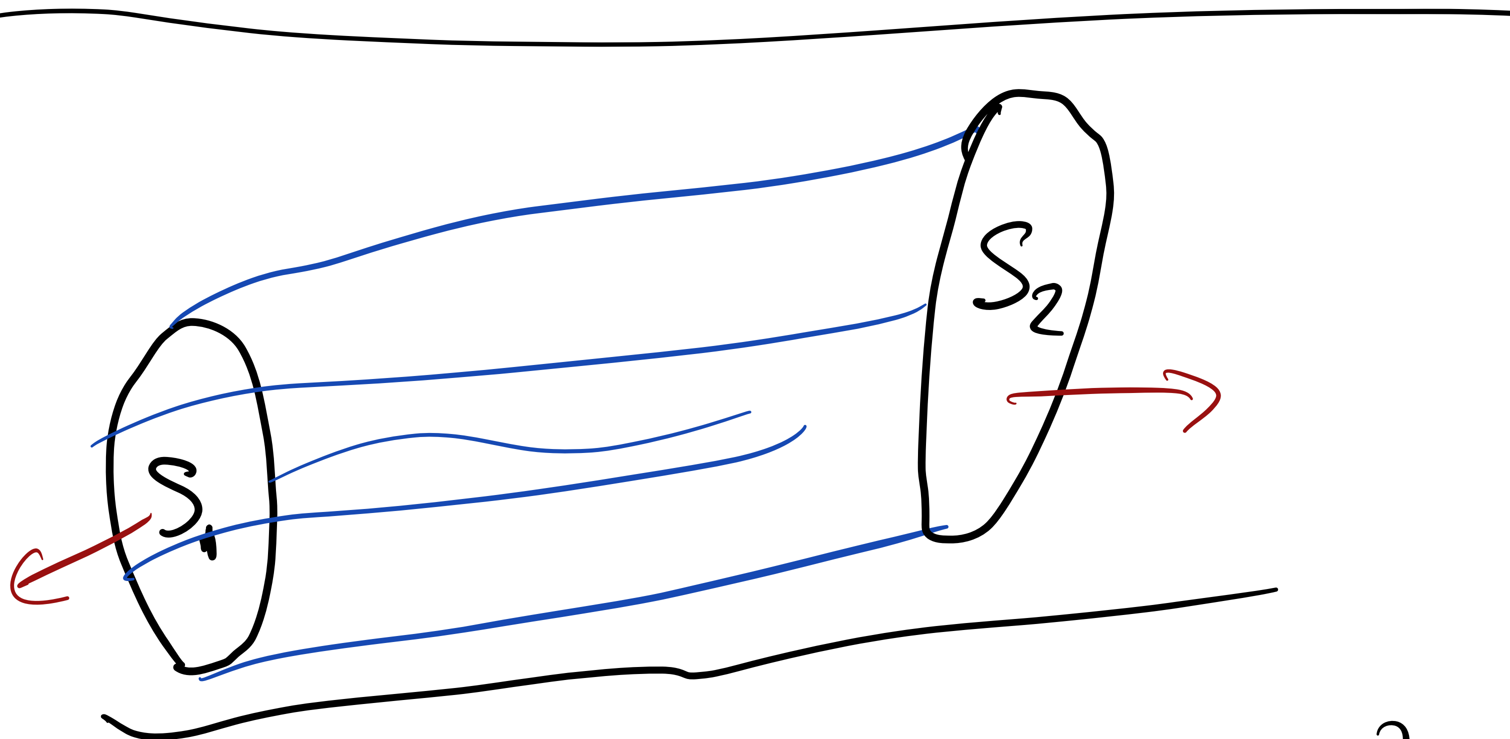
$\omega$: closed 2-form.

$S_1 \subseteq M$, embedded.

Choose a flow $\phi^t: M \to M$

$$S_2 := \phi^{t=1}(S_1)$$

$$\int_{S_1} \omega \overset{?}{=} \int_{S_2} \omega$$




$$\underbrace{C := \left\{ \phi^t(S_1) \,\middle|\, 0 \le t \le 1 \right\}}_{\text{3-mfd in } M}$$

$$\int_{S_1} \omega = \int_{S_2} \omega$$

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$$0 = \int_{C} d\omega = \underbrace{\int_{\partial C} \omega}_{\displaystyle\int_{S_1} \omega - \int_{S_2} \omega}$$

Ex3. $M$: $\begin{cases} \text{cpct} \\ \text{orientable} \\ \dim = n \end{cases}$

$\Omega^k(M) := \{\text{diff forms}\}$

$B^k := \{\text{exact diff. forms}\}$

$Z^k := \{\text{closed diff. forms}\}$

1. $B_k, Z_k \leq \Omega^k$

$0 \in \ldots$

closed under $+$

closed under scalar mult.

d is linear.

2. $\begin{cases} B_k \subset Z_k \\ B_k \neq Z_k \end{cases}$

$\omega \in B_k$ if $\exists \eta \in \Omega^{k-1}$

s.t $\underline{\omega} = \underline{d\eta}$ (exact)

W.T.S $(d\omega = 0)$

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$$\eta = \sum_{i_1 < \dots < i_N} f_{i_1, \dots, i_N} \, dx_{i_1} \wedge \dots \wedge dx_{i_N}$$

$$\omega = d\eta = \sum_j \sum_{i_1 < \dots < i_N} \frac{\partial f_{\dots}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge \dots$$

$$d\omega = d^2\eta = \underbrace{\sum_k \sum_j}_{\text{go through every possible index}} \sum_{i_1<\dots<i_\dots} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge \dots$$

so each $\frac{\partial^2 f}{\partial x_j \partial x_k} dx_j \wedge dx_k$ cancels out with $\frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j$

(Since $dx_j \wedge dx_k = -dx_k \wedge dx_j$, and second derivatives commute when $f$ is smooth (Schwarz))

$$= 0.$$

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3. $Z^k = \{\text{closed } k\text{-forms}\}$

$B^k = \{\text{exact } k\text{-forms}\}$

real vec. spaces (see Arthur's 26/10/24 email)

$$\boxed{H^k := Z_k / B_k}$$

$\Rightarrow$ real vector space

Let $n := \dim M$.

Show:

$$\int : H^n \longrightarrow \mathbb{R}$$

is an isomorphism.

Fix only true if M is connected. Else, RHS is $\mathbb{R}^d$ where $d = \# \left\{ \begin{matrix} \text{connected} \\ \text{components} \\ \text{of } M \end{matrix} \right\}$

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- well-defined:

$$H^n = Z^n / B^n$$

Need to show: $\displaystyle\int \big|_{B_n} \equiv 0$

in other words: let $\eta$ be exact. $\displaystyle\int_{\Pi} \eta = 0$

- $\Pi$: cpct, or$^{ed}$, without bdry
- $\eta = d\chi$

$$\int_M \eta = \int_M d\chi$$

$$= \int_{\underbrace{\partial M}_{=\,\emptyset}} \chi = 0$$

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Integral of an exact form over a mfd w/o bdry is 0

$\circ$ $\int : H^n \longrightarrow \mathbb{R}$

is an isomorphism.

Pf 1:

surjective

+

$\underbrace{\text{injective}}_{\text{hard}}$

$\dim \Lambda^k V = \binom{n}{k}$

Pf 2:

- surjective.
- $H^n = Z^n / B^n$

By ex 1,

$\dim_{\mathbb{R}} H^n \leq 1$

So injectivity is automatic.

So how do we prove:

$$\int : H^n \to \mathbb{R}$$

surjective ?

↳ find $\omega$ s.t $\int \omega \neq 0$.

- $M$ orientable

$\Rightarrow \exists\, \omega$ top-degree

$$\int \omega > 0$$

✓

- $\Lambda^n V \to \dim = n$

$$H^n = Z^n / B^n$$

$$\dim(H^n) \leq 1$$

Have a surjection

$$H^n \to \mathbb{R}$$

$\square$

# Poincaré's lemma:

any closed form on $\mathbb{R}^n$ is exact

(On any mfd, $\exists\ \mathcal{U} \underset{\text{open}}{\subseteq} M$

s.t all closed forms on

$\mathcal{U}$ are exact). Pf: Lee's intro to smooth mfds

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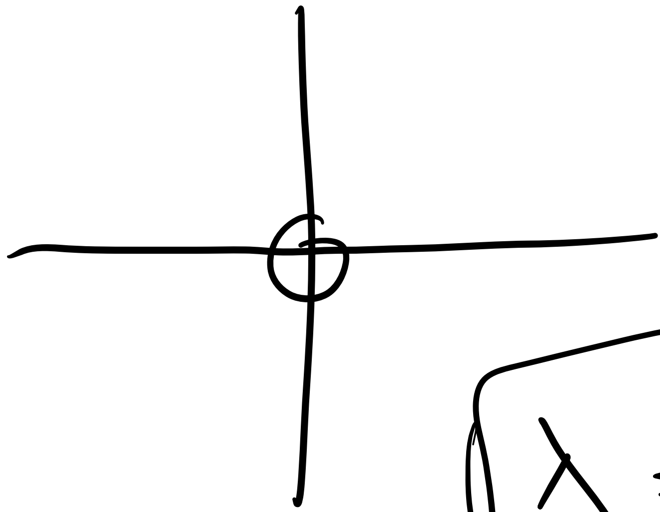
TS $\exists$ forms

$\begin{cases} \text{closed} \\ \text{non-exact} \end{cases}$ on $M$

$\Rightarrow M$ has non-trivial topology.



$\mathbb{R}^2 \setminus \{(0,0)\}$

$$\boxed{\lambda = \frac{x\,dy - y\,dx}{x^2 + y^2}}$$

$d\lambda = 0$

Non-exact?

If $\lambda$ were exact, w'd have

$$\int_{\mathbb{R}^2 \setminus 0} \lambda = 0$$

not true

## Ex. 1

1. $\Lambda^k V^* := \{ \text{linear } k\text{-forms} \}$

$$\text{Basis} = \left\{ \sum_{i_1 < \dots < i_k} \eta_{i_1} \wedge \dots \wedge \eta_{i_k} \;\middle|\; \begin{array}{l} i_1 < \dots < i_k \\ \text{multi-index} \\ \eta: \text{basis of} \\ \text{1-forms for} \\ V^* \end{array} \right.$$

(dualize a basis of V)

2. $K = n$

$$\text{Basis} = \{ dx_1 \wedge \dots \wedge dx_n \}$$

$$= \{ \det \}$$

$\Lambda^n V^*$ : 1-dimensional.

If we can find $v_1, \dots, v_n$

s.t $\det(v_1, \dots, v_n) \neq 0$

Then $\Lambda^n V^* = \mathrm{Span}_{\mathbb{R}}(\det)$

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Ex. 4

Diff 2-form is <u>symplectic</u> if

- $\omega$ is closed ($d\omega = 0$)
- non-degenerate

Show:

S orientable surface

$\Leftrightarrow$ S is symplectic

1. S orient.

$\Rightarrow \exists$ volume form $\omega_{vol}$, $\int_S \omega_{vol} \neq 0$.

$\Rightarrow \omega$ is non-degenerate

(If $\omega$ were degenerate,
there would exist some vector

$X$ s.t $\omega(X, \cdot) \equiv 0$)

- $\omega$ : 2-form on a 2-fold

$d\omega = 0$

2. Show the symplectic structure on $S$ orientable (comm.) is unique (in a reasonable sense)

$\Lambda^2 T^* S \to$ 1-dimensional

any 2-form is the same (up to a cst) $\square$

5. $G$ Lie group :

$\begin{cases} \text{Group } G \\ \text{Also a mfd, where the} \\ \text{group laws are smooth} \end{cases}$

Lie algebra of $G$:

$\mathfrak{g} := \mathrm{Lie}(G)$

$= \{ \underline{\text{Left-inv. vector}} \text{ fields} \}$

$X$ is left-inv

if $(L_g)_* X = X$

where $L_g : G \to G : h \mapsto gh$

Show: $\mathfrak{g} = T_e G$

where $e \in G$

identity

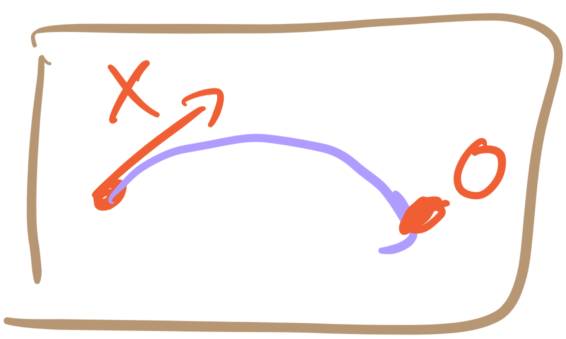
1. W.T.S $\exists$ iso

$$\boxed{\phi : \mathfrak{g} \longrightarrow T_e G}$$

$$X \longmapsto X|_e$$

- injectivity: follows from left invariance.

$$\phi(X) = 0$$



• surjectivity :

Start from $v \in T_e G$.

Want to define $X$ left-inv.

$$\phi(X) = v$$

$$v = X_{e}$$

$X_v$ vector field on $G$

$$\boxed{\begin{aligned} X_v(g) &:= (L_g)_{*e}\, v \\ &= d(L_g)_e\, v \end{aligned}}$$

• smooth: ✓

• left-inv.

$\forall g' \in G$:

$$\underbrace{(L_{g'})_* X_v}\ \overset{\text{want}}{=}\ X_v$$

$$\underbrace{(L_{g'})_*\big|_{pt} (L_g)_*\big|_e \, v}$$

$$\underbrace{d(L_{g'})\big|_{pt} \circ d(L_g)\big|_e \, v}$$

$$\|$$

$\| \quad$ (chain rule)

$$d(L_{g'} \circ L_g)|_e$$

$$L_g : \begin{array}{ccccc} G & \longrightarrow & G & \xrightarrow{L_{g'}} & G \\ h & \longmapsto & gh & \longrightarrow & g'gh \end{array}$$

By chain rule, $X_v$ is left-inv.

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$$\phi(X_v) = v$$

$$|$$

$$\begin{cases} X_v := d(L_g)_e\, v \\ \phi : \mathfrak{g} \longrightarrow T_e G \\ \qquad X \longmapsto X|_e \end{cases}$$

$$\phi(X_v) = d(\mathrm{id})_e\, v$$

$$= \mathrm{id} \cdot v$$

$$= v \qquad \square$$

eg $O(n) := \{A \in M_n \mid A^t A = I\}$

eg Why is it a LG?

(check that everything is smooth).

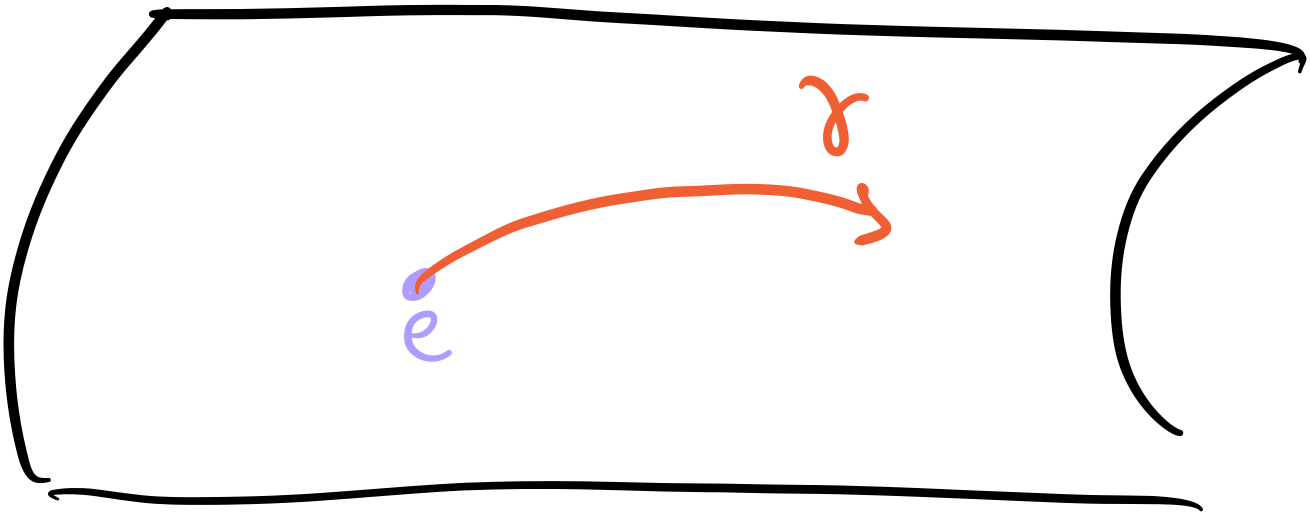
(or just observe $O(n) \subset GL_n$)

What is $\mathrm{Lie}(G)$?

$$\|$$

$$T_e G$$

$G = O(n)$




Path in $O(n)$ starting at $e$

$\gamma : [0,1] \to O(n)$

$\gamma(0) = e$

& matrix

$\gamma(t) = P(t)$

By def$^n$, $P^T(t)\,P(t) = e$

$$\mathrm{Lie}(G) = \{\text{tgt vectors at } e\}$$

$$= \left\{\begin{array}{c}\text{equivalence class} \\ \text{of a path in } G\end{array}\right\}$$

$$\gamma(t) = P(t), \quad \underbrace{P^T P = e}$$

differentiate

$$\underbrace{\dot{P}^T P + P^T \dot{P} = 0}$$

still simplify

$\gamma(0) = e$

$t=0$

$$\boxed{\dot{p}^T + \dot{p} = 0}$$

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$B \in \mathrm{Lie}(G)$

$$\Rightarrow \underbrace{B^T = -B}_{\text{skew-symmetric matrices}}$$

For the other direction, need the fact that

$$\underbrace{\exp : \mathfrak{g} \longrightarrow G}$$

defines local coordinates on $G$.

(See next sheet)